Optimal Design of a Micro-Electrical-Mechanical Systems Actuator

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1 Introduction

Fundamental to the design of an inkjet printer is precise delivery of ink from the printer to the paper. One proposed method is to manufacture a tiny beam of metal in such a way that when one end is heated, the beam bends thereby projecting a tiny volume of ink onto the paper.

A preliminary beam has been manufactured at Eastman Kodak with the overall dimensions $100\mu\text{m} \times 20\mu\text{m} \times 5\mu\text{m}$. This particular beam consisted of two materials, aluminum (Al) and silicon dioxide (SiO_2) in a ratio of 3:2. A voltage pulse of 10μs was applied to the beam heating it up to about 400K and resulting in a maximum rate of deflection of about 0.2ms^{-1}.

The problem set forth was to first model the beam described above in the hopes of understanding the underlying physics. The second goal was to generalize the model to design a beam with perhaps more layers that achieves a maximum deflection rate of at least 1ms^{-1}. Because of the nature of the fluid, the temperature of the beam must not exceed about 400K. In addition, the overall dimensions of the beam are required to be about the same as the preliminary beam discussed above. As a result, the only free parameters are the choice of materials for the beam and in which amounts they should be chosen.

2 Statement of the Problem

Solving the equations for the full beam/fluid flow, even numerically, is a formidable task, and now we shall proceed to simplify the model as much as possible. Of course, we intend to justify this process in the subsequent analysis.
The assumptions:

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- We may treat the problem with one space dimension. Moreover, we shall neglect calculation of the flow field and opt to model the effect of the fluid on the beam with a parameterization scheme.
- The flow carries little fluid away (relative to the length scale of the beam) and so the convective term in temperature conservation equation may be dropped.
- We assume each layer of the beam to be homogeneous and the heating to be uniform; consequently, we expect a uniform temperature profile. Furthermore, we assume linear elasticity theory is sufficient to model the beam, and that boundary conditions may be applied at the initial (unstretched/contracted) positions. We shall also neglect the thermal expansion of the oxide.

These simplifications are implicit in what follows.

3 Heat Transport in the System

Of central importance in the modelling of this problem is the transport of heat from the Al into the SiO_2 and surrounding Isopar fluid. A current is supplied to the aluminum, which generates an amount of heat. Since the thermal expansion coefficient of Al is large with respect to SiO_2, the beam will bend. If we could determine the temperature of the aluminum as a function of time, we could approximate the displacement of the end of the beam and thus estimate the beam speed.

Listed below are some of the thermal properties of of Al, SiO_2 and the surrounding Isopar fluid. The density of a material is denoted as ρ while the specific heat and conductivity are denoted as c_v and k respectively.

Material	ρ ($\mathrm{g\,cm^{-3}}$)	c_v ($\mathrm{J\,g^{-1}K^{-1}}$)	k ($\mathrm{J\,cm^{-1}s^{-1}K^{-1}}$)
Fluid (Isopar)	0.77	2.1	1×10^{-3}
Silicon Dioxide	3.4	0.7	1.38×10^{-2}
Aluminum	2.7	0.5	2.31

With these values, the first question that we ask is can we disregard temperature variations in the oxide? If the temperature variations in the oxide layer are negligible, then thermally, we could simply model the beam as being made out of aluminum. The rule of thumb is that in time Δt, heat diffuses a length Δx given by the expression $\Delta x = (k\Delta t/\rho c_v)^{1/2}$. Hence for SiO_2, heat diffuses a length approximately 1μm in 5μs. Since the depth of the SiO_2 is approximately 3μm, we cannot disregard the temperature variations in SiO_2. Therefore, we must account for both materials.

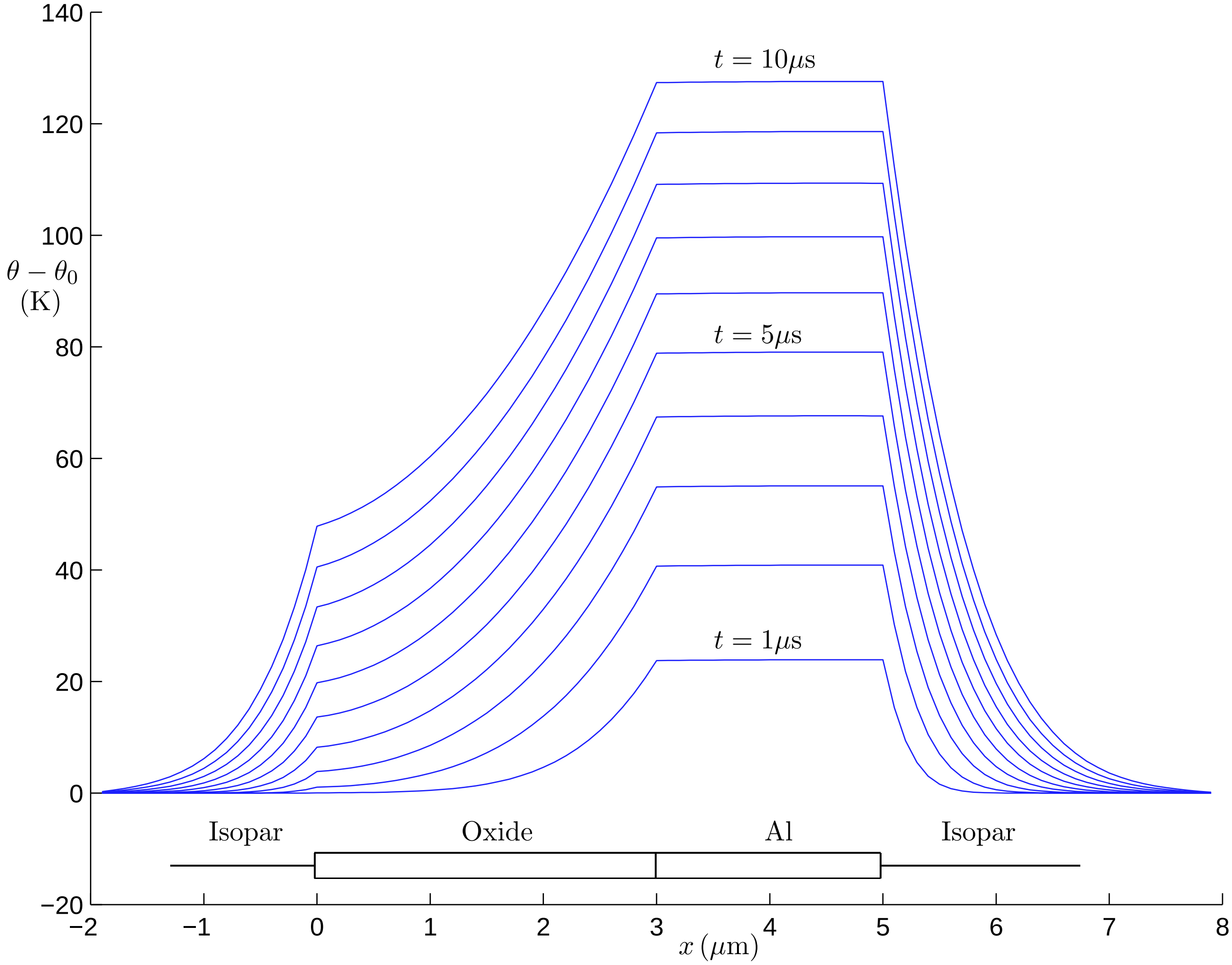

Figure 1: Temperature through a cross section of the beam as heat is applied.

The equations governing the heat flow are:

$$\begin{aligned}
\rho_f c_{v\,f} \theta_t &= k_f \theta_{xx} && \text{(Fluid)} \\
\rho_{\text{ox}} c_{v\,\text{ox}} \theta_t &= k_{\text{ox}} \theta_{xx} && \text{(Silicon Oxide)} \\
\rho_{\text{Al}} c_{v\,\text{Al}} \theta_t &= k_{\text{Al}} \theta_{xx} + Q && \text{(Aluminum)}
\end{aligned} \tag{1}$$

where $Q = 5.35 \times 10^7$ Watts cm^{-3}. The boundary conditions are determined by the empirical fact that temperature is continuous and energy is conserved across the interface boundaries. These conditions imply

$$\begin{aligned}
\theta(\text{interface}^-) &= \theta(\text{interface}^+) \\
k^- \theta_x(\text{interface}^-) &= k^+ \theta_x(\text{interface}^+)
\end{aligned} \tag{2}$$

at any interface. In addition, at infinity the system should be at room temperature so that $\theta(x = \pm\infty, t) = \theta_0 = 300\text{K}$.

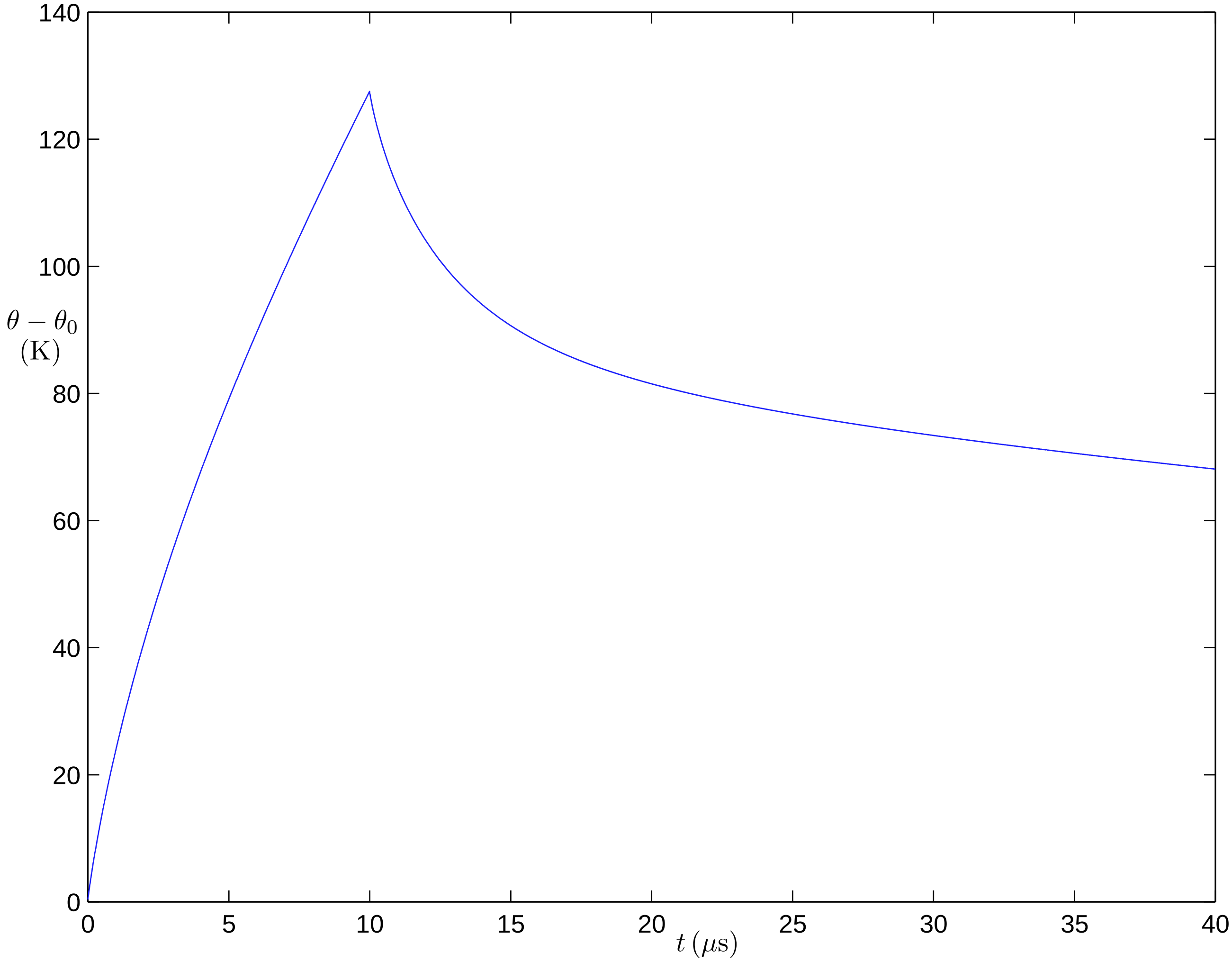

Figure 2: Temperature of the aluminum layer for a $t = 10\mu$s heating pulse.

We decided to limit our scope to the one dimensional problem. Notice that the one dimensional heat flow equation in aluminum can be greatly simplified by integrating. As a result, we get

$$\rho_{\mathrm{Al}} c_{v\,\mathrm{Al}} \bar{\theta}_t = \frac{k_f \theta_x \Big|_c - k_{\mathrm{ox}} \theta_x \Big|_b}{L_{\mathrm{Al}}} + Q \qquad (3)$$

where $\bar{\theta}_t$ is the rate of change of the average temperature of aluminum, L_{Al} is the width of the aluminum, and b and c denote boundaries oxide/aluminum and aluminum/fluid respectively. Since the conductivity k of aluminum is so high, it can be assumed that the temperature variation across the aluminum is zero. Hence the temperature of aluminum is spatially uniform. This fact greatly improves the efficiency of our numerical schemes. The numerical scheme we use did not make use of equation (3) but our results justify this approximation. The resulting temperature profile is displayed in figures 1 and 2.

4 Modelling the Beam

For the beam we consider a laminated beam with N layers labelled $\{1, 2, \ldots, N\}$ where layer j has a Young's modulus of E_j, a density of ρ_j and a thickness of $h_j - h_{j-1}$. With this notation, we take $h_0 = 0$ and $h_N = H$ the overall height of the laminated beam. When the beam is bent the surface outside the curve is stretched while the surface inside the curve is compressed. Internal to the beam there must be some surface which is neither stretched or compressed. This surface is known as the *neutral* surface. The location of this neutral surface, y_0, is found by summing the stress (force per unit area) in each of the layers and noting that the resultant stress is zero. This procedure gives

$$y_0 = \frac{\sum_{j=1}^{N} E_j(h_j^2 - h_{j-1}^2)}{2\sum_{j=1}^{N} E_j(h_j - h_{j-1})}. \tag{4}$$

It is interesting to note that if the Young's modulus was the same for all of the N layers then the above expression becomes a telescoping series and the neutral surface would lie at $h_N/2 = H/2$ which is the height of the centre of mass if the layers also all have the same density.

4.1 Beam Equation

Having located the neutral surface, one can determine the beam equation for this laminated structure. This is accomplished by computing the moment in each of the N layers at two horizontal positions, $x = x_0$ and $x = x_0 + \Delta x$. The details of this derivation are simple yet tedious. The resulting beam equation is

$$\rho H u_{tt} + D u_{xxxx} = P$$

where

$$\rho H = \sum_{j=1}^{N} \rho_j (h_j - h_{j-1}), \qquad D = \frac{1}{3}\sum_{j=1}^{N} E_j (y - y_0)^3 \Big|_{h_{j-1}}^{h_j} \tag{5}$$

are the weighted density and the composite flexural rigidity respectively. and P is the external pressure. If $E_j = E\,\forall j$ then using (4) we find $D = EH^3/12$ as one would expect for a uniform beam of thickness H. The value of E and α for the various materials are listed below.

Material	E ($\mathrm{g\,cm^{-1}s^{-2}}$)	α ($\mathrm{K^{-1}}$)
Silicon Dioxide	6×10^{11}	$\simeq 0$
Aluminum	20×10^{11}	16×10^{-6}

4.2 Boundary Conditions

In order to be well posed, the equation for the beam requires a number of boundary conditions and initial conditions. The initial conditions are simply that the beam has no velocity and is not bent. That is, $u(x,0) = 0 = u_t(x,0)$.

There are four boundary conditions. Since the beam is fixed and clamped at the origin $x = 0$ we easily identify the conditions $u(0,t) = 0$ and $u_x(0,t) = 0$. In addition, the free end, $x = L$, does not experience any shear stress and as such, $u_{xxx}(L,t) = 0$.

The fourth boundary condition arises from the application of heat. Since the beam is laminated, each of the layers will expand at different rates when heated. This imbalance in the strains of the various layers creates a moment at the end $x = L$. We derive this temperature dependent moment next.

We first recall that the stress and strain are related by

$$\frac{F_j}{A_j} = E_j \frac{\Delta l_j}{l_j} \tag{6}$$

where E_j is the Young's modulus of the jth layer. Therefore a layer with $A_j = W(h_j - h_{j-1})$ will have $F_j = E_j W(h_j - h_{j-1})\Delta l_j / l_j$. The magnitude of l_j will depend on the layer. Before any heating takes place, each of the layers has a length denoted as l_0 and if we now heat the beam, each of the layers expands at a different rate. Let α_j denote the expansion rate of the jth layer so that $l_j = (1 + \alpha_j \theta) l_0$ is the amount the jth layer would have expanded at the temperature θ if it was not connected to the other layers. If we set l to be the mean amount of expansion of the beam as a whole after the various layers have expanded we have for the jth layer that

$$F_j = E_j W(h_j - h_{j-1}) \frac{l - l_j}{l_j}.$$

However, these individual forces must cancel out so that $\sum_{j=1}^{N} F_j = 0$. Solving for l gives

$$l = \frac{\displaystyle\sum_{j=1}^{N} E_j (h_j - h_{j-1})}{\displaystyle\sum_{j=1}^{N} \frac{E_j}{l_j} (h_j - h_{j-1})}. \tag{7}$$

The quantity of interest is the the ratio $(l - l_j)/l_j$ and using the fact that even for temperatures on the order of 400K, $\alpha_j\theta \ll 1$ so using (7) gives the approximation

$$\frac{F_j}{A_j} = E_j \frac{l - l_j}{l_j} \simeq \theta E_j(\bar{\alpha} - \alpha_j) \qquad \text{where} \qquad \bar{\alpha} = \frac{\sum_{j=1}^{N} E_j \alpha_j (h_j - h_{j-1})}{\sum_{j=1}^{N} E_j (h_j - h_{j-1})}. \tag{8}$$

The moment generated by each layer satisfies $\theta E_j(\bar{\alpha} - \alpha_j) = E_j(y - y_0)u_{xx}(L)$. Multiplying by a factor of $(y - y_0)$ and integrating over the layers, one finds the total effective moment at the point $x = L$ to be

$$u_{xx}(L, t) = \frac{\theta(t)}{2D} \sum_{j=1}^{N} E_j(\bar{\alpha} - \alpha_j)(y - y_0)^2 \Big|_{h_{j-1}}^{h_j} = \Gamma\theta(t) \tag{9}$$

which is linear with respect to the applied temperature.

5 Beam Fluid Interaction

Consider the following version of the beam equation that accounts to at least a first order approximation, for both the drag and the viscosity of the fluid

$$(\beta + \rho H)u_{tt} = -Du_{xxxx} - ku_t. \tag{10}$$

An expression for the natural frequency of the beam can be obtained by using separation of variables. Let $u(x, t) = F(x)G(t)$ and consider a slightly simplified version of the boundary conditions where the beam is not heated

$$u(x, 0) = u_t(x, 0) = u(0, t) = u_x(0, t) = u_{xx}(L, t) = u_{xxx}(L, t) = 0.$$

Under the separation, one obtains two expressions. For the spatial variable

$$F^{iv} - \frac{\lambda^4}{D}F = 0; \qquad F(0) = F'(0) = F''(L) = F'''(L) = 0$$

and for the temporal variable

$$(\beta + \rho H)G'' + kG' + \lambda^4 G = 0; \qquad G(0) = G'(0) = 0. \tag{11}$$

Focusing on the spatial equation, we find that

$$F(x) = A\left[\sin\left(\frac{\lambda x}{D^{1/4}}\right) - \sinh\left(\frac{\lambda x}{D^{1/4}}\right)\right] + B\left[\cos\left(\frac{\lambda x}{D^{1/4}}\right) - \cosh\left(\frac{\lambda x}{D^{1/4}}\right)\right]$$

where A and B are constants. The eigenvalues for λ arise from the boundary conditions at $x = L$. Computing the second and third derivatives at L leads to the compatibility condition

$$\begin{vmatrix} -\cos\xi - \cosh\xi & \sin\xi - \sinh\xi \\ -\sin\xi - \sinh\xi & -\cos\xi - \cosh\xi \end{vmatrix} = 0 \qquad \text{with} \qquad \xi = \frac{\lambda L}{D^{1/4}}.$$

This implies that the eigenvalues satisfy $1+\cos\xi\cosh\xi = 0$ whose solutions are given by $\xi_0 = \pm 1.8751$ and $\xi_n \simeq \pm(2n+1)\pi/2$ for $n \in \mathbb{N}$. The fundamental frequency and damping of the beam can now be determined by looking at the temporal equation.

5.1 Determining β and k

We observe from the experimental data available that, throughout its motion, the beam oscillates about some varying mean deflection. Not only this, but it is clear that, once the heat supply to the beam is turned off, the amplitude of these oscillations in the fluid decreases in time. Thus, into our model, we incorporate terms associated with a damped harmonic oscillator system, which will model the effect of the viscous fluid on the motion of the beam.

Since our model is one dimensional, we shall consider the free end of the beam, oscillating in one dimension in the fluid, as analogous to the mass in a mass-spring-dashpot system. For a mass m, attached to the free end of a spring with spring constant c, and moving in a dashpot containing fluid with damping coefficient $2b$, the motion of the mass is governed by

$$m\ddot{x} + 2b\dot{x} + cx = 0. \tag{12}$$

Oscillatory solutions of this equation have the form

$$x(t) = Ae^{-bt/m}\sin\left(\sqrt{\frac{mc-b^2}{m^2}}t\right);$$

we identify the frequency of oscillation as $(mc-b^2)^{1/2}/m$ and the decay rate as b/m. Now, the frequency of oscillations in fluid appears constant, and was measured as 3.45×10^5 Hz. The appropriate data for frequency and damping calculations is summarized in the table below.

Fluid	Fundamental Frequency (MHz)	Amplitude at 15 μs (μm)	Amplitude at 35 μs (μm)
Air	0.484	0.27	0.26
Isopar	0.345	0.293	0.086

The decay rate, measured over the remaining time after 20μs, is $b/m = 6.13 \times 10^4$. Following the separation of variables method we choose the fundamental mode $c = \lambda_0^4$, and so

$$\frac{\lambda_0^4}{m} - (6.13 \times 10^4)^2 = 4\pi^2 f^2_{\text{Isopar}} = 4.70 \times 10^{12},$$

where $\lambda_0 = \xi_0^4 D/L^4$. That is,

$$m = \frac{\xi_0^4 D}{4.70 \times 10^{12} L^4} = 2.63 \times 10^{-12} \frac{D}{L^4} \quad \text{and} \quad b = 1.61 \times 10^{-7} \frac{D}{L^4}.$$

A comparison of (12) with the separation of variables (11) method yields

$$\beta = 2.63 \times 10^{-12} \frac{D}{L^4} - \rho H, \qquad k = 3.22 \times 10^{-7} \frac{D}{L^4} \tag{13}$$

as first approximations for the constants to be used in to match the given data. These numbers are later tuned to match the data as close as possible.

6 Results

As there were two goals in this project two cases were considered. The first case was a beam in the ratio of 2:3 of Al to SiO_2 in the Invar fluid. While in the second case, a ratio of 1:2 was chosen to maximize the coupling moment induced by the temperature. In this second case the beam is slightly thinner and therefore gets hotter for the same amount of energy input. The paramenters for these two cases are summarized below. $T_{\text{max}} = 397.3$K in case 2:3 and 402.4K in case 1:2 respectively.

Parameter	Case 2:3	Case 1:2
ρH	1.56×10^{-3}	9.50×10^{-4}
D	10.442	2.2542
Γ	4.755×10^{-2}	7.985×10^{-2}
Q	4.08×10^7	6.80×10^7
β	1.32×10^{-3}	1.32×10^{-3}
k	123	73.8

The values of β and k are determined by matching the solution to the given experimental data in the case 2:3. Once these values are known, the same value of β is used in the case 1:2 as the same volume if Invar fluid is being accelerated in both cases. The value of k scales with the thickness of the beam.

For a given geometry a solution of the heat equation (1-2) determines $\theta(t)$. This time dependent temperature is then applied as a boundary condition for the beam equation (5), (8-10). Numerical solutions for the two cases are plotted below along with the experimental points. The agreement is astounding.

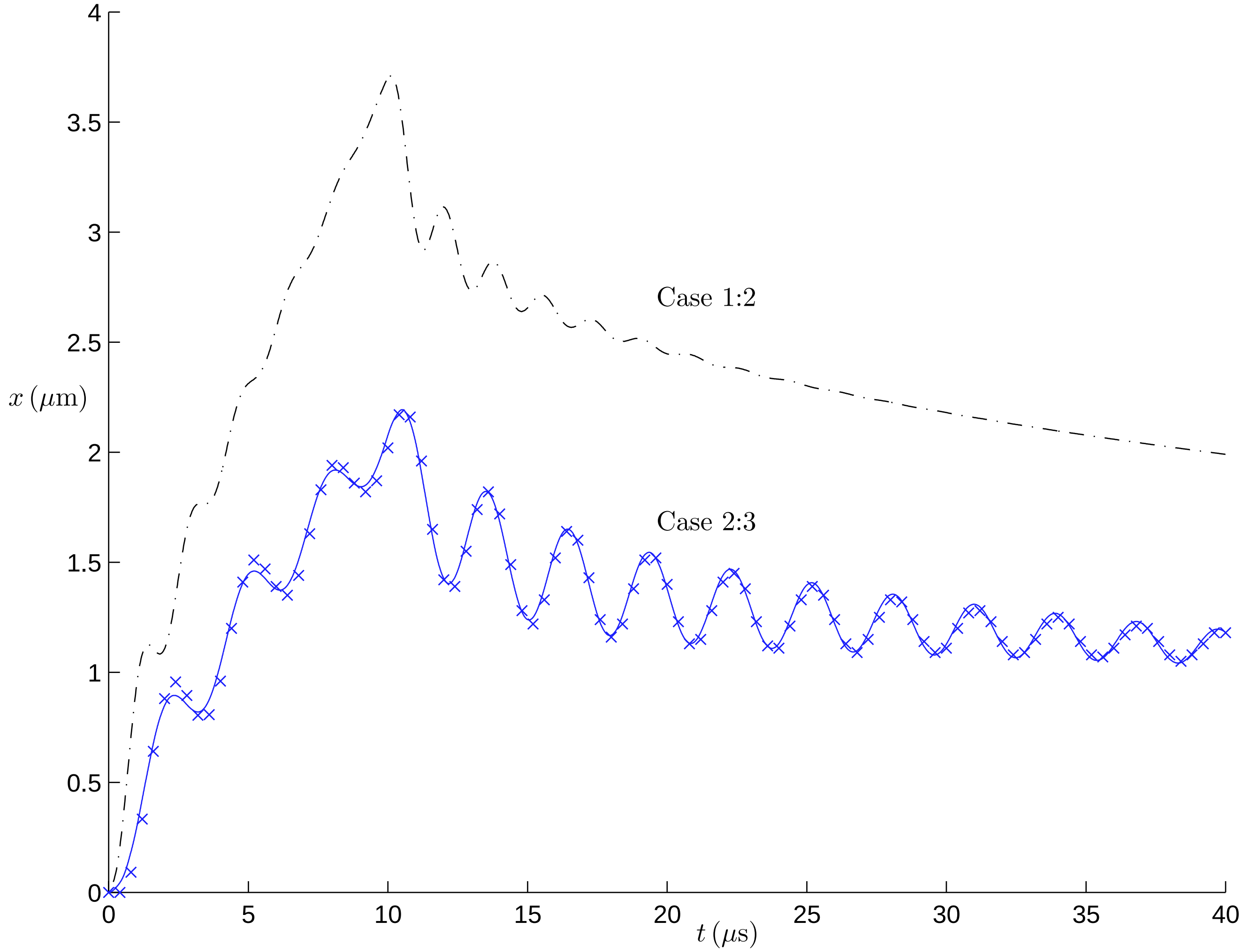

Figure 3: Position of the end of the beam with respect to time

7 Conclusions and Directions

Our initial goal was to accelerate Isopar fluid to a speed of 1ms^{-1} over $10\mu s$ using a beam that deflects when heated. Our first objective was to develop an appropriate physical model for the problem. The key simplifying assumptions included treating the problem as one dimensional, relying on the linear beam equation and neglecting the details of the fluid flow.

We were able to reproduce experimental results with high agreement. Furthermore, applying the theory, we were able to improve the speed of the fluid by a factor of 2. Although we did not obtain our objective, we did make significant progress. The next step would be to consider more than two layers and possibly different materials. Despite the inherent difficulties, studying the two dimensional problem would be of interest. There's also evidence that an insulating layer would increase speed; this may increase the relaxation time beyond acceptable limits.