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# Newtonian limit of axially symmetric spacetimes

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We illustrate how the Ehlers' formal mathematical definition of the Newtonian limit requires additional information to successfully determine the Newtonian limit. This information can be obtained through the physical arguments of Cooperstock's definition. We show that Ehlers' formalism is equivalent to Cooperstock's derivation of the Newtonian limit when the physical arguments are included in the former definition.

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Many papers have been written on how to calculate relativistic covariant multipole moments for axially symmetric space-times [1–7]. In a recent paper [8], Quevedo uses Ehlers' [9] definition to calculate the Newtonian multipole moments in the Weyl class of axially symmetric space-times. There is a set of parameters $\{q_n\}$ which are important in determining the Newtonian limit. The formal mathematical definition presented by Ehlers does not explicitly take into account the inherent structure of the $\{q_n\}$, which is revealed only through physical arguments. The physical motivation has been discussed by Cooperstock [10] in his method for determining the Newtonian limit of an axially symmetric spacetime. Quevedo discusses the fact that $\{q_n\}$ must be bounded above in order for the Newtonian multipole expansion to converge. An explicit form for these bounds are found in Cooperstock's method. Finally, we show that Cooperstock's and Ehlers' methods are equivalent when proper consideration are given to $\{q_n\}$.

Ehlers' definition of the Newtonian limit is

$$\Phi = \lim_{\lambda\to 0} \frac{1}{\lambda}\psi(\rho,z,\lambda)\,, \tag{1}$$

where $\lambda = c^{-2}$ ($c$ = speed of light) and $\psi(\rho,z,\lambda)$ is the metric function $\psi$ in Weyl's canonical coordinates containing the parameter $\lambda$. Specifically, Weyl's metric is [11]

$$ds^2 = e^{2\psi}dt^2 - e^{2\gamma-2\psi}(d\rho^2+dz^2) - e^{-2\psi}\rho^2 d\phi^2\,, \tag{2}$$

where $\psi$, $\gamma$ are functions of $\rho$ and $z$.

The asymptotically flat solution $\psi$ of the Einstein field equations (EFEs) in Weyl canonical coordinates is

$$\psi = \sum_{n=0}^{\infty}(-1)^{n+1} q_n P_n(y) Q_n(x)\,, \tag{3}$$

where

$$x = \frac{r_+ + r_-}{2GM\lambda}\,, y = \frac{r_+ - r_-}{2GM\lambda}\,,$$

and

$$r_\pm = \sqrt{\rho^2 + (z \pm GM\lambda)^2}\,.$$

Expressing $Q_n(x)$ as a descending power series in $x$ [12],

$$Q_n(x) = 2^n \sum_{s=0}^{\infty} b(n,s) x^{-(n+1+2s)} \tag{4}$$

with

$$b(n,s) \equiv \frac{(n+s)!\,(n+2s)!}{s!\,(2n+2s+1)!}, \tag{5}$$

one can rewrite Eq. (3) as

$$\psi(\lambda) = \sum_{n=0}^{\infty}\psi_n = \sum_{n=0}^{\infty}(-1)^{n+1} q_n P_n\left(\frac{r_+ - r_-}{2GM\lambda}\right)2^n \times \sum_{s=0}^{\infty} b(n,s)\left(\frac{r_+ + r_-}{2GM\lambda}\right)^{-(n+2s+1)}. \tag{6}$$

It can be shown that [8]

$$\lim_{\lambda\to 0} P_n\left(\frac{r_+ - r_-}{2GM\lambda}\right) = P_n\left(\frac{z}{\sqrt{\rho^2+z^2}}\right). \tag{7}$$

Hence, from Eq. (1),

$$\Phi = \sum_{n=0}^{\infty}(-1)^{n+1} P_n\left(\frac{z}{\sqrt{\rho^2+z^2}}\right) \times 2^n\left[\lim_{\lambda\to 0}\frac{q_n}{\lambda}\sum_{s=0}^{\infty} b(n,s)\left(\frac{2GM\lambda}{r_+ + r_-}\right)^{n+2s+1}\right]. \tag{8}$$

This equation is equivalent to Eq. (52) of Ref. [8], except that in Ref. [8], the coefficient $q_n$ is placed outside of the limit. This implies that $q_n$ is independent of $\lambda$. The $\lambda$-dependence is subsequently introduced into $q_n$ through the new parameter $\tilde{q}_n = q_n(G\lambda)^n$ after the limit has presumably been taken. It would have been appropriate to make the substitution

$$q_n(\lambda) = \frac{\tilde{q}_n}{(G\lambda)^n} \tag{9}$$

where it is explicitly stated that $\tilde{q}_n$ is $\lambda$ independent.

Thus, substituting Eq. (9) into Eq. (8) and taking the limit, one obtains

$$\Phi = G\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n!}{(2n+1)!!}\tilde{q}_n M^{n+1}\frac{P_n(\cos\theta)}{r^{n+1}},$$
$$\cos\theta = \frac{z}{r}, \qquad r \equiv \sqrt{\rho^2+z^2} \tag{10}$$

which is Quevedo's Eq. (53). One may ask this point why $\{q_n\}$ were chosen in the form of Eq.(9). One *cannot* obtain this information from the *definition* of Ehlers. It must be added through a physical argument.

We shall now examine the physical reason behind choosing $q_n(\lambda)$ of the form of Eq. (9). It is based on Cooperstock's method [10] for determining the Newtonian limit of an axially symmetric source. We shall utilize Cooperstock's arguments within Ehlers' framework to work out a slightly different form for $q_n$ than that of Eq. (9) and show that this form is the upper bound for the $\{q_n\}$ required in lemma 4 of Ref. [8]. It should be noted that Cooperstock used geometrized units ($G = c = 1$) in his derivations. To utilize Ehlers' definition, our derivation will keep $G$ and $\lambda = c^{-2}$.

Cooperstock defines a "nearly Newtonian source" as being a source which satisfies the condition

$$\frac{GM\lambda}{L} \ll 1, \tag{11}$$

where $L$ is the characteristic size of the source. It is also stated that the characteristic size of the Newtonian $2^n$-pole multipoles are $ML^n$, $n = 0, 1, \ldots$ . Now the Newtonian potential has the form

$$\Phi = G\sum_{n=0}^{\infty}N_n\frac{P_n(\cos\theta)}{r^{n+1}}, \tag{12}$$

which implies that each $N_n$ of Eq. (12) must be bounded above by

$$N_n \sim ML^n\,. \tag{13}$$

If we examine the "Newtonian-like" terms of Eq. (8), it becomes apparent that

$$q_n \sim \left(\frac{L}{GM\lambda}\right)^n \tag{14}$$

is the required upper bound on $\{q_n\}$. In this process, one is essentially working backwards from the expected form of the Newtonian potential. Cooperstock's introduction of the characteristic size of the source $L$ is the key to placing an upper limit on the value of the multipole moments. If we make the substitution

$$q_n = \tilde{q}_n\left(\frac{L}{GM\lambda}\right)^n \tag{15}$$

where the $\tilde{q}_n$'s are dimensionless, $\lambda$ independent and of order 1, into Eq. (8) and then apply the limit $\lambda \to 0$, the resulting equation is

$$\Phi = GM\sum_{n=0}^{\infty}(-1)^{n+1}\frac{n!}{(2n+1)!!}\tilde{q}_n L^n\frac{P_n(\cos\theta)}{r^{n+1}}. \tag{16}$$

Therefore it is concluded that Eq. (16) represents the Newtonian limit only when all the expansion coefficients are bounded through Eq. (14). Furthermore, the multipole moments are identified as

$$N_n = (-1)^{n+1}\frac{n!}{(2n+1)!!}\tilde{q}_n ML^n. \tag{17}$$

This result differs from

$$N_n = (-1)^{n+1}\frac{n!}{(2n+1)!!}\tilde{q}_n M^{n+1}, \tag{18}$$

which was derived in Ref. [8]. We know from Eq. (12) that the $n$th Newtonian multipole moment should have dimensions [mass]·[length]$^n$. Therefore, the revised $\tilde{q}_n$ of Eq. (9) implicitly has dimensions

$$\tilde{q}_n \sim \frac{[\text{length}]^n}{[\text{mass}]^n}. \tag{19}$$

By utilizing Cooperstock's physical arguments, the order of magnitude of this length, i.e., the characteristic size of the source, becomes apparent.

It is easy to show that Ehlers' definition for determining the Newtonian limit is actually equivalent to Cooperstock's method when proper consideration is given to $\{q_n\}$. In his procedure, Cooperstock transforms $\psi(\rho, z)$ into spherical polar coordinates $(r, \theta)$ through the relations

$$x = \frac{r}{GM\lambda} - 1, \quad y = \cos\theta, \tag{20}$$

and then asymptotically expands $g_{tt} = e^{2\psi}$ about $r \to \infty$. Cooperstock identifies the function $f(r, \theta, \lambda)$ by setting

$$e^{2\psi(r,\theta,\lambda)} = 1 + 2\,f(r,\theta,\lambda)\,, \tag{21}$$

where $f(r, \theta, \lambda)$ is not the Newtonian potential as one might expect. [Cooperstock uses geometrized units where the $\lambda$ dependence is hidden. Note that $r = GM\lambda + (r_+ + r_-)/2$. Therefore $r^2 = \rho^2 + z^2$ *only* in the limit $\lambda \to 0$.] Erez and Rosen [13] interpreted $f(r, \theta, \lambda)$ as the Newtonian potential. Cooperstock expressed the sentiment that no Newtonian limit considerations had been applied to the source. Quevedo noted that his result [Eq. (10)] contradicted that of Erez and Rosen. The logical conclusion one would draw from the apparent contradiction is that the condition $r \to \infty$ is insufficient for determining the Newtonian potential for an axially symmetric source with multipole moments. It is the approximation procedures in combination with the structure of the $\{q_n\}$, which must be taken into consideration in order to obtain the correct Newtonian multipole structure.

In order to show the equivalence of Cooperstock's and Ehlers' methods, we will show that $f(r,\theta,\lambda)$ can be substituted for $\psi(\rho,z,\lambda)$ in Ehlers' definition and that the limiting procedure $\lambda \to 0$ is equivalent to Cooperstock's approximation $m/L \ll 1$.

Theorem 10 of Ref. [9] shows that

$$g_{tt} = e^{2\lambda U}, \tag{22}$$

where $U(\rho,z)$ is independent of $\lambda$, and states that the Newtonian limit is $\Phi = U|_{\lambda=0}$. Comparing Eq. (22) with Eq. (2) we note

$$\psi = \lambda U. \tag{23}$$

Ehlers also indicated that in order for $g_{tt}$ to make sense *as a Newtonian limit* one must be able to write

$$\begin{aligned} g_{tt} &= 1 + 2\lambda U + \mathrm{O}(\lambda^2) \\ &= 1 + 2h(\lambda U). \end{aligned} \tag{24}$$

Using L'Hôpital's rule, one obtains

$$\begin{aligned} \lim_{\lambda\to 0} \frac{1}{\lambda} h(\psi) &= \lim_{\lambda\to 0} \frac{1}{\lambda} h(\lambda U) = \lim_{\lambda\to 0} h'(\lambda U) U \\ &= h'(0)\Phi = \Phi. \end{aligned} \tag{25}$$

Hence

$$\lim_{\lambda\to 0} \frac{1}{\lambda}\psi = \lim_{\lambda\to 0} \frac{1}{\lambda} h(\psi). \tag{26}$$

However, $\psi(\rho,z,\lambda) = \psi(r,\theta,\lambda)$ through the transformation equations (20). [Once $\psi(\rho,z)$ is identified through Weyl's metric, the function $\psi$ may be expressed in any coordinate system such as $(x,y)$ or $(r,\theta)$.] Comparing Eq. (21) to Eq. (24) implies

$$h(\psi(\rho,z,\lambda)) = h(\psi(r,\theta,\lambda)) = f(r,\theta,\lambda). \tag{27}$$

Hence we must conclude

$$\lim_{\lambda\to 0} \frac{1}{\lambda} f(r,\theta,\lambda) = \Phi. \tag{28}$$

Therefore, we have shown that $f(r,\theta,\lambda)$ is a valid substitution for $\psi(\rho,z)$ in Ehlers' definition. We should emphasize that the coordinates [$\rho, z$ in (25) and $r,\theta$ in (28)] are held constant when taking the limit and that the equality between (28) and (25) has been shown to hold for the coordinate transformation of Eq. (20) only. It is not necessarily true for a general $\lambda$-dependent transformation.

Cooperstock used Eq. (14) and (11) in geometrized units to eliminate the "cross terms" on the basis that they are negligible compared with the Newtonian terms. However, Ehlers' limit $\lambda \to 0$ implies

$$\frac{GM\lambda}{L} = \frac{m}{L} \ll 1. \tag{29}$$

The cross terms in $f(r,\theta)$, which Cooperstock eliminates through his physical argument are the terms with residual powers of $\lambda$ and are thus eliminated via the limit $\lambda \to 0$ in Ehlers' formalism. In Cooperstock's method one can use any system of units (including geometrized units where the explicit dependence of $G$ and $\lambda$ disappear) to calculate the Newtonian potential. In Ehlers' definition, one must use units which retain $c$ as a parameter so that the limit $c \to \infty$ can be taken.

Thus we have confirmed that the function $f(r,\theta,\lambda)$ is a valid substitution for $\psi(\rho,z,\lambda)$ in Ehlers' limit and that the limiting procedure $\lambda \to 0$ is equivalent to Cooperstock's criterion $m/L \ll 1$. Hence the two methods for calculating the Newtonian potential for an axially symmetric space-time are equivalent when $\{q_n\}$ are examined from a physical point of view. The advantage of Cooperstock's method is that it makes clear the connection between the $\{q_n\}$ and the source of the field, thus simplifying the problem of extracting a Newtonian limit.

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