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Error Estimates for Galerkin Approximations to the Periodic Schrödinger–Poisson System

Wir stellen Konvergenzraten für eine Galerkin approximation für das periodische Schrödinger–Poisson–Problem im Einheitswürfel $[0,1]^3$ *auf. Die Fehler–Abschätzungen transformieren sich in entsprechende* L^∞*- Fehlerabschätzungen für die Wignersche Verteilungsfunktion.*

We establish convergence rates for a Galerkin approximation to the peridic Schrödinger–Poisson problem in the unit cube $[0,1]^3$*. The error estimates transform into corresponding* L^∞*-error estimates for the Wigner distribution function.*

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1 Introduction

We are concerned with the three-dimensional periodic Schrödinger–Poisson problem in the unit cube $Q = [0,1]^3$, this problem has been studied in [1]. The equations are

$$i\psi_{m,t} = -\frac{1}{2}\Delta\psi_m + V(\Psi)\psi_m \tag{1.1.a}$$

$$\Delta V = q(x) - n \tag{1.1.b}$$

$$n = \sum \lambda_m |\psi_m|^2. \tag{1.1.c}$$

For simplicity, we have set $\hbar = 1$ and the particle mass $M = 1$. We also assume initial charge neutrality $\int_Q n(x)\,dx = \int_Q q(x)\,dx$. In the sequel, we confine our discussion to the case $q \equiv 1$, but our method generalizes to any sufficiently smooth, one-periodic q representing the density of positive background charge. We also assume, without restricting the generality, that $\lambda_m > 0$ for all m, and that $\sum \lambda_m = 1$.

The Schrödinger–Poisson system (SP), and its equivalence to the Wigner–Poisson system (WP), are discussed in detail in [2]–[6]. The periodic Vlasov-Poisson is treated in [7] and [8]. In [9], the quantum periodic case is studied.

(1.1a - 1.1c) are complemented by the initial conditions

$$\psi_m(x,0) = \varphi_m(x) \tag{1.1.d}$$

and periodic boundary conditions in Q. The φ_m's form an orthonormal system in $L^2(Q)$.

In [1], it was shown by a suitable Galerkin approximation that Eqns. (1.1a-d) possess a unique, global-in-time, strong solution in the space

$$C([0,T]; X^2).$$

Here, we define

$$X^k = \{\Psi = (\psi_m);\ \psi_m \in H^k_{loc}(\mathbb{R}^3) \wedge \forall x \in Q\ \forall m \in \mathbb{N}\ \forall l \in \mathbb{Z}^3 : \psi_m(x+l) = \psi_m(x) \text{ in the } L^2\text{-sense}\}$$

with the norm

$$\|\Psi\|^2_{X^k} = \sum_{\substack{m\in\mathbb{N} \\ |\alpha|\le k}} \lambda_m \|D^\alpha \psi_m\|^2_{L^2(Q)}.$$

We assume that $\Phi = (\varphi_m) \in X^2$, a necessary assumption for the existence proof of [1]. Instead of X^0 we will just write X.

The objective of this paper is to obtain error estimates for Galerkin approximations of solutions of system (1.1a-d) obtained in [1]. The Galerkin sequence $(\Psi^{(N)})_{N\in\mathbb{N}}$ is there defined by

$$\psi_m^{(N)}(x,t) = \sum_{|k|\le N} d^{(N)}_{m,k}(t) h_k(x), \tag{1.2}$$

where $h_k(x) = e^{2\pi i k\cdot x}$, $k \in \mathbb{Z}^3$, and

$$\left(\left[i\partial_t \psi_m^{(N)}(x,t) + \frac{1}{2}\Delta\psi_m^{(N)} - V^{(N)}(\Psi^{(N)})\psi_m^{(N)}\right], h_k\right) = 0 \tag{1.3}$$

for $|k| \le N$. Here,

$$V^{(N)}(\Psi^{(N)}) = \frac{1}{4\pi^2} \sum_{\substack{|k|\le 2N\\ k\neq 0}} \frac{n_k^{(N)}}{k^2} h_k, \tag{1.4}$$

where we define:

$$n^{(N)}(x,t) = \sum_m \lambda_m |\psi_m^{(N)}(x,t)|^2 = \sum_{|k|\le 2N} n_k^{(N)} h_k(x). \tag{1.5}$$

A short calculation shows

$$n_k^{(N)} = \sum_m \lambda_m \sum_{\substack{|l|\le N\\ |l-k|\le N}} d_{m,l}^{(N)} \overline{d}_{m,l-k}^{(N)}, \text{ where } |k| \le 2N,\ k\neq 0, \text{ and } n_0^{(N)} = 1. \tag{1.6}$$

$V^{(N)}$ is the explicit solution of

$$\Delta V^{(N)} = 1 - n^{(N)} \tag{1.7}$$

with periodic boundary conditions.

From (1.3) and the subsequent formulas one sees that the coefficients $d_{m,k}^{(N)}$ in (1.2) must satisfy the first order differential system

$$\dot{d}_{m,l}^{(N)}(t) = -2\pi^2 i l^2 d_{m,l}^{(N)}(t) - \frac{i}{4\pi^2} \sum_{\substack{0<|k|<2N\\ |l-k|\le N}} \frac{n_k^{(N)}}{k^2} d_{m,l-k}^{(N)}(t), \quad |l| \le N, \tag{1.8}$$

subject to the initial conditions

$$d_{m,l}^{(N)}(0) = (\varphi_m, h_l). \tag{1.9}$$

It is proved in [1] that for each $N \in \mathbb{N}$ the system (1.8-9) has a global unique classical solution such that $\|\Psi^{(N)}(\cdot,t)\|_{X^0} = \|\Phi\|_{X^0} = 1$ (see [1], Lemma 2.1), and that the conservation law

$$\frac{d}{dt}\int_Q \left\{ |\nabla\Psi^{(N)}(t)|^2 + |\nabla V^{(N)}(t)|^2 \right\} dx = 0,$$

where $|\nabla\Psi^{(N)}(t)|^2 \stackrel{\text{def.}}{=} \sum_{m\in\mathbb{N}} \lambda_m |\nabla\psi_m^{(N)}|^2$ holds.

2 Some Auxiliary Results

We begin by proving a generalization of a Lemma in [10] which is crucial for our error estimates. Recall that we are in three dimensions.

Lemma 2.1 *Let $v \in H_p^k(Q)$, $k \in \mathbb{N}$, $k \ge 1$. Then if*

$$\begin{aligned} &v_n = \textstyle\sum_{|j|\le n} (v, h_j) h_j, \qquad h_j = e^{2\pi i j\cdot x},\\ &\|v - v_n\|_{L^2(Q)}^2 \le 3(2\pi)^{-2k} \left(\tfrac{3}{n}\right)^{2k} \|v\|_{H^k(Q)}^2 \end{aligned} \tag{2.1}$$

Proof. Assume $v \in C_p^\infty(Q)$ (p for 1-periodic). By Pythagoras' Theorem,

$$\|v - v_n\|_{L^2(Q)}^2 = \sum_{|j|>n} |(v,h_j)|^2. \tag{2.2}$$

If $|j_1| + |j_2| + |j_3| > n$, there must be one index larger than $n/3$, so the sum on the right of (2.2) is bounded by

$$\sum_{j_2,j_3} \sum_{|j_1|>n/3} |(v,h_j)|^2 \tag{2.3}$$

plus two other sums of the same kind (with $|j_2| > n/3$, $|j_3| > n/3$ respectively). Writing

$$e^{-2\pi i j\cdot x} = \left(\partial_{x_1}^k e^{-2\pi i j\cdot x}\right) \cdot \frac{1}{(-2\pi i j_1)^k},$$

intergrating by parts $k-1$ times, using the periodicity assumption and the continuity, the expression (2.3) can be estimated by

$$\sum_{j_2,j_3}\sum_{|j_1|>n/3}\left|\int_Q \partial^k_{x_1} v(x) e^{-2\pi i j\cdot x}\,dx\right|_2 \frac{1}{|2\pi j_1|^{2k}} \le \frac{1}{\left|2\pi\frac{n}{3}\right|^{2k}}\sum_{|j_1|>n/3}\sum_{j_2,j_3}\left|\left(\partial^k_{x_1} v, h_j\right)\right|^2.$$

The sum, being a sum of squares of Fourier coefficients, is bounded by $\|v\|^2_{H^k(Q)}$. The result follows for $v\in C^\infty_p(Q)$. It extends to $v\in H^k_p(Q)$by a density argument. □

Corollary 2.2 *Let $S=[0,T]$ and $w\in L^2(S,H^k_p(Q))$, such that $D^\alpha w(\cdot,t)$, $|\alpha|\le k-1$, is periodic for all $t\in S$, and let $w_n(\cdot,t)=\sum_{|k|\le n}(w(\cdot,t),h_k)h_k$. Then for $C=3(2\pi)^{-2k}3^{2k}$,*

$$\|w-w_n\|^2_{L^2(S,L^2(Q))}\le \frac{C}{n^{2k}}\|w\|^2_{L^2(S,H^k)}. \tag{2.4}$$

Lemma 2.3 *Let $W^{(N)}(\cdot,t)=\Psi(\cdot,t)-\Psi^{(N)}(\cdot,t)$ and let*

$$\hat\Psi_N=\left(\sum_{|j|\le N}(\psi_m(\cdot,t),h_j)h_j\right)_{m\in\mathbb{N}}$$

be the projection of the true solution onto the span of $\{h_j,|j|\le N\}$. Then there are constants C and C_T (independent of N, but C_T depending on T) such that

$$\frac{d}{dt}\|W^{(N)}(\cdot,t)\|^2_X\le C\|\Psi(\cdot,t)-\hat\Psi_N(\cdot,t)\|_X+C_T\|W^{(N)}(\cdot,t)\|^2_X. \tag{2.5}$$

Proof. Let $D^{(N)}_m=i\psi^{(N)}_{m,t}+\frac12\Delta\psi^{(N)}_m-V(N)(\Psi^{(N)})\psi^{(N)}_m$ (cf. (1.3)). Then, as $\hat\psi_{N,m}-\psi^{(N)}_m\in$ span $\{h_j;\ |j|\le N\}$, by the definition of the Galerkin approximation (1.3) we can write

$$(D^{(N)}_m,\psi_m-\psi^{(N)}_m)=(D^{(N)}_m,\psi_m-\hat\psi_{N,m}+\hat\psi_{N,m}-\psi^{(N)}_m)=(D^{(N)}_m,\psi_m-\hat\psi_{N,m}). \tag{2.6}$$

Let $w^{(N)}_m$ be the m-th component of $W^{(N)}_m$, i.e. $w^{(N)}_m=\psi^{(N)}_m-\psi_m$. Note that $\frac{\partial}{\partial t}\psi^{(N)}_m\in$ span $\{h_k;|k|\le N\}$, because the h_k's are independent of t, and that $\Delta\psi^{(N)}_m\in$ span $\{h_k;|k|\le N\}$, because the h_k's are eigenfunctions of Δ. Also, by construction

$$\psi_m-\hat\psi_{N,m}\in(\mathrm{span}\{h_j;|j|\le N\})^\perp,$$

and therefore, from (2.6)

$$\left(D^{(N)}_m,w^{(N)}_m\right)=\left(-V^{(N)}(\Psi^{(N)})\psi^{(N)}_m,\psi_m-\hat\psi_{N,m}\right). \tag{2.7}$$

Using the fact that $\Psi(x,t)$ is the exact solution of (1.1 a-d), we can rewirte (2.7) as

$$\left(i\psi^{(N)}_{m,t}+\frac12\Delta\psi^{(N)}_m-V^{(N)}(\Psi^{(N)})\psi^{(N)}_m-i\psi_{m,t}-\frac12\Delta\psi_m+V(\Psi)\psi_m,w^{(N)}_m\right)=\left(V^{(N)}(\Psi^{(N)})\psi^{(N)}_m,\psi_m-\hat\psi_{N,m}\right),$$

i.e.,

$$\left(-iw^{(N)}_{m,t}-\frac12\Delta w^{(N)}_m,w^{(N)}_m\right)=\left(V^{(N)}(\Psi^{(N)})\psi^{(N)}_m-V(\Psi)\psi_m,w^{(N)}_m\right)-\left(V^{(N)}(\psi^{(N)})\psi^{(N)}_m,\psi_m-\hat\psi_{N,m}\right). \tag{2.8}$$

Now

$$\frac{d}{dt}\left|w^{(N)}_m\right|^2_{L^2}=2\,\mathrm{Re}\int\overline{w^{(N)}_m}w^{(N)}_{m,t}\,dx=2\,\mathrm{Im}\int\overline{w^{(N)}_m}iw^{(N)}_{m,t}\,dx.$$

As $\mathrm{Im}\int\Delta w^{(N)}_m\cdot w^{(N)}_m\,dx=0$, it follows from the previous identity (2.8) that

$$\frac{d}{dt}\|w^{(N)}_m\|^2_{L^2}=2\,\mathrm{Im}\int\left(V(\Psi)\psi_m-V^{(N)}(\Psi^{(N)})\psi^{(N)}_m\right)\overline{w^{(N)}_m}\,dx+2\,\mathrm{Im}\int V^{(N)}(\Psi^{(N)})\psi^{(N)}_m\overline{(\psi_m-\hat\psi_{N,m})}\,dx,$$

i.e.

$$\begin{aligned}\frac{d}{dt}\|w^{(N)}_m\|^2_{L^2}&=2\,\mathrm{Im}\int\left(V(\Psi)-V^{(N)}(\Psi^{(N)})\right)\psi_m\overline{w^{(N)}_m}\,dx+2\,\mathrm{Im}\int V^{(N)}(\Psi^{(N)})w^{(N)}_m\cdot\overline{w^{(N)}_m}\,dx\\&\quad+2\,\mathrm{Im}\int V^{(N)}(\Psi^{(N)})\psi^{(N)}_m\overline{(\psi_m-\hat\psi_{N,m})}\,dx.\end{aligned} \tag{2.9}$$

The second term on the right vanishes as $V^{(N)}(\Psi^{(N)})$ is real (see equation (1.7)).

We remark that $\|V^{(N)}(\Psi^{(N)})\|_{L^\infty} \leq C$, where C depends only on $\|\Phi\|_X$ (see [1], Lemma 2.3). The third term can therefore be estimated by

$$C\|\psi_m^{(N)}\|_{L^2}\|\psi_m - \hat{\psi}_{N,M}\|_{L^2},$$

(C is a generic constant) and as

$$\|\psi_m^{(N)}(\cdot,t)\|_{L^2} = \|\psi_m^{(N)}(\cdot,0)\|_{L^2} = \left(\sum_{|j|\leq N} |(\varphi_m, h_j)|^2\right)^{1/2} \leq \|\varphi_m\|_{L^2} = 1,$$

we get a bound

$$C\|\psi_m - \hat{\psi}_{N,m}\|_{L^2}$$

for this term.

To estimate the first term on the right of (2.9), note that by (1.4)

$$\|V(\Psi) - V^{(N)}(\Psi^{(N)})\|_{L^\infty} \leq C \sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}} \frac{|n_k - n_k^{(N)}|}{k^2}$$

where n_k is defined by

$$n(x,t) = \sum_{k\in\mathbb{Z}^3} n_k(t) h_k.$$

Thus

$$\begin{aligned}
\|V(\Psi) - V^{(N)}(\Psi^{(N)})\|_{L^\infty} &\leq C\left(\sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}} \frac{1}{(k^2)^2}\right)^{1/2} \left(\sum_{k\in\mathbb{Z}^3} |n_k - n_k^{(N)}|^2\right)^{1/2} \\
&\leq C\left(\sum_{k\in\mathbb{Z}^3} |n_k - n_k^{(N)}|^2\right)^{1/2} = C\|n - n^{(N)}\|^2_{L^2} \\
&= C\sqrt{\int_Q |\sum \lambda_r(|\psi_r|^2 - |\psi_r^{(N)}|^2)|^2\,dx} \\
&= C\sqrt{\int_Q |\sum \lambda_r((\psi_r - \psi_r^{(N)})\overline{\psi_r} + (\overline{\psi_r} - \overline{\psi_r^{(N)}})\psi_r^{(N)})|^2\,dx} \\
&\leq C\left(\int_Q (\sum \lambda_r|\psi_r|^2)(\sum \lambda_r|\psi_r - \psi_r^{(N)}|^2)\,dx\right)^{1/2} \\
&\quad + C\left(\int_Q (\sum \lambda_r|\psi_r^{(N)}|^2)(\sum \lambda_r|\psi_r - \psi_r^{(N)}|^2)\,dx\right)^{1/2} \\
&\leq C_T\|\Psi - \Psi^{(N)}\|_X,
\end{aligned}$$

as

$$n = \sum \lambda_r|\psi_r|^2 \text{ and } n^{(N)} = \sum \lambda_r|\psi_r^{(N)}|^2$$

are both bounded in the L^∞ -norm on $[0,T]$. See [1], Lemma 3.2.

Summarizing, we have proven that

$$\begin{aligned}
\sum \lambda_m \left|\int (V(\Psi) - V^{(N)}(\Psi^{(N)}))\psi_m \overline{w_m^{(N)}}\,dx\right| &\leq C_T\|\Psi - \Psi^{(N)}\|_X \sum_m \int |\lambda_m \psi_m \overline{w_m^{(N)}}\,dx \\
&\leq C_T\|\Psi - \Psi^{(N)}\|_X \|W^{(N)}\|_X = C_T\|W^{(N)}\|_X^2,
\end{aligned}$$

and as

$$\sum \lambda_m \|\psi_m - \hat{\psi}_{N,m}\|_{L^2} \leq C\|\Psi - \hat{\Psi}_N\|_X,$$

it follows that

$$\frac{d}{dt}\|W^{(N)}(t,\cdot)\|_X^2 = \frac{d}{dt}\sum_m \lambda_m \|w_m^{(N)}(t,\cdot)\|^2_{L^2} \leq C\|\Psi - \hat{\Psi}_N\|_X + C_T\|W^{(N)}(t,\cdot)\|_X^2.$$

This completes the proof of Lemma 2.3. □

3 The Main Theorem

We are now ready for the main result of this paper.

Theorem 3.1 *If $\Phi \in X^2$, then the Galerkin sequence $\Psi^{(N)}$ associated with Φ as initial value (see (1.9)) satisfies, on $[0,T]$, the error estimates*

$$\|\Psi - \Psi^{(N)}\|_{L^\infty([0,T];X)} \leq \frac{C_T}{N},$$

where C_T depends on T and on $\|\Phi\|_{X^2}$.

Proof. We denote generic constants by C and generic constants that depend on T by C_T. Note that $\Phi^{(N)} = \hat{\Phi}_N$, i.e., the Fourier series for Φ gives the initial data for the Galerkin approximation. From Lemma 2.3,

$$\|W^{(N)}(\cdot,t)\|_X^2 \leq \|W^{(N)}(\cdot,0)\|_X^2 + C\int_0^t \|\Psi(\cdot,\tau) - \hat{\Psi}_N(\cdot,\tau)\|_X \, d\tau + C_T \int_0^t \|W^{(N)}(\cdot,\tau)\|_X^2 \, d\tau.$$

By Lemma 2.1,

$$\|W^{(N)}(\cdot,0)\|_X^2 \leq \frac{C}{N^4}\|\Phi\|_{X^2}^2$$

and

$$C\int_0^t \|\Psi(\cdot,\tau) - \hat{\Psi}_N(\cdot,\tau)\|_X \, d\tau \leq \frac{C}{N^2}\int_0^t \|\Psi(\cdot,\tau)\|_{X^2} \, d\tau \leq \frac{C_T}{N^2}$$

(because $\|\Psi(\cdot,t)\|_{X^2}$ is uniformly bounded on $[0,T)$, see [1]). So

$$\|W^{(N)}(\cdot,t)\|_X^2 \leq \frac{C}{N^4}\|\Phi\|_{X^2}^2 + \frac{C_T}{N^2} + C_T\int_0^t \|W^{(N)}(\cdot,\tau)\|_X^2 \, d\tau,$$

and by Gronwall's Lemma

$$\|W^{(N)}(\cdot,t)\|_X^2 \leq \frac{C_T}{N^2}(1 + \|\Phi\|_{X^2}^2)e^{C_T t}.$$

This completes the proof. □

Now we look at the periodic Wigner–Poisson problem (see [1]) which is the system of equations

$$\partial_t \rho_{w,k} + v_k \nabla_x \rho_{w,k} - i\Theta(V)\rho_{w,k} = 0, \tag{3.1}$$

$$\Delta V = 1 - n(x,t), \tag{3.2}$$

$$\rho_{w,k}(x,0) = \rho_{w,k,I}(x), \tag{3.3}$$

where $(\rho_{w,k})$ is the sequence of Wigner functions given by $\rho_w(x, v_k, t)$, $v_k = 2\pi k$, $k \in \mathbb{Z}^3$, $\Theta(V)$ is the pseudo-differential operator

$$\begin{aligned}\Theta(V)\rho_{w,k} &= \sum_{k'}\int_{Q'}\left[V\left(x+\frac{\eta}{2},t\right) - V\left(x-\frac{\eta}{2},t\right)\right]\rho_{w,k'}(x,t)\cdot e^{2\pi i(k-k')}\, d\eta \\ &= \frac{1}{8}\sum_m \lambda_m \int_{Q'} e^{2\pi i k\cdot z}\psi_m\left(x-\frac{z}{2},t\right)\overline{\psi_m}\left(x+\frac{z}{2},t\right)\left[V\left(x+\frac{z}{2},t\right) - V\left(x-\frac{z}{2},t\right)\right] dz,\end{aligned}$$

$Q' = [-1,1]^3$. Here we have used that $\rho_{w,k}(x,t)$ is given by the Wigner transform

$$\rho_{w,k}(x,t) = \frac{1}{8}\sum_m \lambda_m \int_{Q'} e^{2\pi i k\cdot z}\psi_m\left(x-\frac{z}{2},t\right)\overline{\psi_m}\left(x+\frac{z}{2},t\right) dz. \tag{3.4}$$

We consider the Wigner transform of the Galerkin sequence

$$\rho_{w,k}^{(N)}(x,t) = \frac{1}{8}\sum_m \lambda_m \int_{Q'} e^{2\pi i k\cdot z}\psi_m^{(N)}\left(x-\frac{z}{2},t\right)\overline{\psi_m^{(N)}}\left(x+\frac{z}{2},t\right) dz. \tag{3.5}$$

This can be computed explicitly to give

$$\rho_{w,k}^{(N)}(x,t) = \sum \lambda_m \sum_{|l|\leq N, |2k-1|\leq N} d_{l,m}^{(N)}\overline{d}_{2k-1,m}^{(N)}(t)e^{4\pi i x(l-k)}, \tag{3.6}$$

where $d_{l,m}^{(N)}(t)$ are the coefficients of the Galerkin approximation $\psi_m^{(N)}$. Finally, we write

$$\begin{aligned}
&\rho_{w,k}(x,t) - \rho_{w,k}^{(N)}(x,t)\\
&= \frac{1}{8}\sum \lambda_m \int_{Q'} \left(\psi_m\left(x-\frac{z}{2},t\right)\overline{\psi}_m\left(x+\frac{z}{2},t\right) - \psi_m^{(N)}\left(x-\frac{z}{2},t\right)\overline{\psi}_m^{(N)}\left(x+\frac{z}{2},t\right)\right) e^{2\pi i k\cdot z}\,dz\\
&= \frac{1}{8}\sum \lambda_m \int_{Q'} \left\{\left(\psi_m\left(x-\frac{z}{2},t\right) - \psi_m^{(N)}\left(x-\frac{z}{2},t\right)\right)\overline{\psi}_m\left(x+\frac{z}{2},t\right)\right.\\
&\qquad \left. + \left(\overline{\psi}_m\left(x+\frac{z}{2},t\right) - \overline{\psi}_m^{(N)}\left(x+\frac{z}{2},t\right)\right)\psi_m^{(N)}\left(x-\frac{z}{2},t\right)\right\} e^{2\pi i k\cdot z}\,dz. \qquad (3.7)
\end{aligned}$$

By the Cauchy–Schwarz inequality and periodicity we have

$$\begin{aligned}
&\int_{Q'} \left|\left(\psi_m\left(x-\frac{z}{2},t\right) - \psi_m^{(N)}\left(x-\frac{z}{2},t\right)\right)\overline{\psi}_m\left(x+\frac{z}{2},t\right)\right| dz\\
&\sqrt{8}\|\psi_m(\cdot,t) - \psi_m^{(N)}(\cdot,t)\|_{L^2(Q)}\|\psi_m\|_{L^2(Q)} \qquad (3.8)
\end{aligned}$$

and a similar estimate for the other term in the right hand side of (3.7). By using Theorem 3.1 and (3.8) we arrive at

$$\|\rho_{w,k} - \rho_{w,k}^{(N)}\|_{L^\infty(Q\times S)} \leq \frac{C_T}{N}$$

with the same constant C_T as in Theorem 3.1 and $S = [0,T]$.

Let us assume now that the initial function of the Wigner–Poisson system (3.1)-(3.3) is the Wigner transform of a suitable initial function $\Phi \in X^2$ of the SP system (1.1a)-(1.1d). Let us call this assumption condition (C).

It is well known (see [1]) that the solution of the Wigner–Poisson system (3.1)-(3.3) is given by (3.4), where $\Psi = (\psi_m)$ is the unique strong 1-periodic solution of (1.1a)-(1.1d). Thus we have proved

Theorem 3.2 *Let condition (C) be satisfied for the initial data $(\rho_{w,k,I})$. Then for the Wigner transform $\rho_{w,k}^{(N)}$ of the Galerkin sequence $(\Psi^{(N)})$ we have*

$$\|\rho_{w,k} - \rho_{w,k}^{(N)}\|_{L^\infty(Q\times S)} \leq \frac{C_T}{N}$$

for all $k \in \mathbb{N}$, $N \in \mathbb{N}$, where $(\rho_{w,k})$ is the unique 1-periodic solution sequence of the WP system, and C_T C_T is the constant from Theorem 3.1.

Remark: If the solution Ψ of the SP system (1.1a)-(1.1d) is actually a strong X^K-solution, then one can prove similarly the error estimate

$$\|\Psi - \Psi^{(N)}\|_{L^\infty([0,T],X^k)} \leq \frac{C_T}{N^{k/2}}.$$

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