ANALYSIS OF HOPF/HOPF BIFURCATIONS IN NONLOCAL HYPERBOLIC MODELS FOR SELF-ORGANISED AGGREGATIONS

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The modelling and investigation of complex spatial and spatio-temporal patterns exhibited by a various self-organised biological aggregations has become one of the most rapidly-expanding research areas. Generally, the majority of the studies in this area either try to reproduce numerically the observed patterns, or use existence results to prove analytically that the models can exhibit certain types of patterns. Here, we focus on a class of nonlocal hyperbolic models for self-organised movement and aggregations, and investigate the bifurcation of some spatial and spatio-temporal patterns observed numerically near a codimension-2 Hopf/Hopf bifurcation point. Using weakly nonlinear analysis and the symmetry of the model, we identify analytically all types of solutions that can exist in the neighbourhood of this bifurcation point. We also discuss the stability of these solutions, and the implication of these stability results on the observed numerical patterns.

Keywords: Nonlocal hyperbolic model; self-organised aggregations; bifurcation and symmetry.

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1. Introduction

The investigation of self-organised biological aggregations represents a research area that has undergone a great expansion in the past years. The main reason for this

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rapid expansion is represented by the complex spatial and spatio-temporal patterns exhibited by various biological aggregations (from milling or travelling schools of fish, to zigzagging flocks of birds or rippling behaviour in bacterial colonies). The mathematical models derived to investigate these behaviours try to propose plausible biological mechanisms regarding the interactions at the microscale level (e.g. inter-individual repulsion and attraction, individual velocities, or individual interactions with the environment) that could explain the observed behaviours/patterns at the macroscale level (i.e. the shape, size and dynamics of the group). To this end, the mathematical models either: (i) focus mainly on numerical simulations with the purpose of reproducing and comparing the simulated aggregation patterns with the available data (and thus ascertaining the correctness of the microscale-level assumptions incorporated into the models), or (ii) focus mainly on the analytical results (e.g. show the existence of particular solutions exhibited by these models, or try to connect biological interactions at the microscale and macroscale levels). The first approach can be found in many individual-based models, which sometimes are too complex to be investigated analytically. The second approach can be found in kinetic models (sometimes derived from individual-based models), for which the numerical simulations can become very complicated. While there are models for self-organised aggregations that combine both numerical and analytical results, usually these models do not investigate the bifurcation dynamics of the resulting patterns. Such an investigation is important from both a mathematical and a biological point of view, since it can help us in understanding the factors (parameters) that can cause the transitions between different group behaviours. It can also help us in understanding what kind of patterns (behaviours) we would expect to see bifurcating from certain other patterns (behaviours). Among the few studies that investigate the bifurcation of spatial and spatio-temporal patterns as various parameters (e.g. population size or magnitude of social interactions) are changed, we remark Topaz et al. and Eftimie et al. Both studies focus on codimension-1 bifurcations. In particular, Topaz et al. focuses on a steady-state (Turing) bifurcation in a nonlocal parabolic equation, and shows that the aggregations (stationary pulses) bifurcate sub-critically. The study in Eftimie et al. on nonlocal hyperbolic models for aggregation and movement for left and right moving densities, focuses on the bifurcation dynamics of stationary pulses (via a steady-state bifurcation) and travelling trains (via a Hopf bifurcation). These two patterns bifurcate also sub-critically.

The class of nonlocal hyperbolic models introduced in Eftimie et al. can exhibit more exotic patterns: travelling (rotating) waves, standing waves, modulated waves, and even zigzagging patterns and breather patterns (see Eftimie et al.). Here we focus on a particular sub-model introduced in Eftimie et al. (sub-model M4), which was shown to exhibit all these exotic patterns. The spatio-temporal regularity of these patterns indicates the presence of symmetries. In fact, all sub-models (M1–M5) introduced in Eftimie et al. which have periodic boundary conditions, are actually O(2)-symmetric (where the group O(2) contains a
subgroup of rotations — $\text{SO}(2)$ — and a reflection symmetry which interchanges left and right moving densities along with a reflection in the domain. Since many of the patterns exhibited by these nonlocal hyperbolic models (i.e. travelling waves, standing waves and modulated waves) seem to occur near $\text{O}(2)$ Hopf/Hopf bifurcations, in this paper we will investigate in more detail this type of codimension-2 bifurcation.

While there are many studies on codimension-2 bifurcations in various physics and engineering models, to our knowledge there are almost no such studies in models for self-organised biological aggregations. The standard methods for investigating codimension-2 Hopf/Hopf bifurcations are centre manifold reductions or weakly nonlinear analysis and multiple scales. In this paper, we build upon the analysis performed in Eftimie et al., and focus on weakly nonlinear analysis. However, to make sure that we are not missing any solutions, we combine this analysis with the investigation of model symmetries. The unfolding analysis of the $\text{O}(2)$ Hopf/Hopf bifurcation, in particular the classification of solutions according to their symmetry groups, follows the classifications in Chossat et al. and Golubitsky et al.

We start in Sec. 2 by describing briefly the nonlocal hyperbolic model. We also discuss the symmetries of this model and perform a linear analysis near a homogeneous equilibrium using group-theoretic methods. Using the expressions for the solutions of the linear system, we obtain contour plots for these solutions and compare them with the results of the numerical simulations performed in the neighbourhood of the Hopf/Hopf point. Then, in Sec. 3, we perform a weakly nonlinear analysis of the system and classify the steady states of the resulting amplitude equations according to the symmetry classification. We also investigate the stability of these amplitude steady states and use the results to explain the numerical simulations. We conclude in Sec. 4 with a general discussion and a summary of the results.

2. Model Description

The following hyperbolic model was proposed in Eftimie et al. to describe the evolution of densities of right-moving ($u^+$) and left-moving ($u^-$) individuals:

\[
\begin{align*}
\partial_t u^+(x, t) + \partial_x (\gamma u^+(x, t)) &= -\lambda^+ [u^+, u^-] u^+(x, t) + \lambda^- [u^+, u^-] u^-(x, t), \\
\partial_t u^-(x, t) - \partial_x (\gamma u^-(x, t)) &= \lambda^+ [u^+, u^-] u^+(x, t) - \lambda^- [u^+, u^-] u^-(x, t), \\
u^\pm(x, 0) &= u^\pm_0(x).
\end{align*}
\]

Here, $\gamma$ is the constant speed, while $\lambda^\pm$ are functionals that describe the turning behaviour of individuals, in response to their neighbours:

\[
\lambda^\pm [u^+, u^-] = \lambda_1 + \lambda_2 f(y^\pm [u^+, u^-] - y^\pm_0 [u^+, u^-] + y^\pm_1 [u^+, u^-])
\]

\[= (\lambda_1 + \lambda_2 f(0)) + \lambda_2 (f(y^\pm_0 - y^\pm_1) + f(y^\pm_1) - f(0)).\]

The turning function $f$ is positive, increasing and bounded: $f(u) = 0.5 + 0.5 \tanh(u - 2)$. The terms $\lambda_1 + \lambda_2 f(0)$ and $\lambda_2 (f(y^\pm_0) - f(0))$ describe the baseline...
random turning rate and the bias turning rate, respectively. As \( f \) is chosen such
that \( f(0) \ll 1 \), we can approximate the random turning by \( \lambda_1 \) and the directed
turning — in response to neighbours’ density and movement direction — by \( \lambda_2 f(y^\pm) \).
These assumptions for the turning rates are biologically realistic, since, as noted by
Lotka,\textsuperscript{25} “the type of motion presented by living organisms ... can be regarded as
containing both a systematically directed and also a random element”. Finally, the
terms \( y^r \), \( y^a \) and \( y^al \) describe the repulsive, attractive and alignment social inter-
actions.\textsuperscript{18} Since individuals in an ecological aggregation can interact — via visual,
sound or olfactory signals — with neighbours positioned further away,\textsuperscript{26} we assume
that these social interactions are nonlocal. Throughout this paper, we assume that
a reference individual positioned at \((x, t)\) interacts socially only with its
neighbours moving towards it (see Fig. 1). This could be the case when individuals
use directional signals to communicate with each other (e.g. tactile signals or
directional sound signals\textsuperscript{18}). This situation corresponds to model M4 introduced in
Eftimie et al.\textsuperscript{18} The diagram in Fig. 1 can be translated into mathematical equations
as follows:
\begin{align*}
y^r_{r,a} &= q_r a \int_0^\infty K_{r,a}(s)(u^\mp(x \pm s, t) - u^\pm(x \mp s, t))ds, \quad (2.3a) \\
y^a_{al} &= q_al \int_0^\infty K_{al}(s)(u^\mp(x \pm s, t) - u^\pm(x \mp s, t))ds. \quad (2.3b)
\end{align*}
Here, \( q_r \), \( q_a \) and \( q_al \) are the magnitudes of the repulsive, attractive and alignment
social interactions. The kernels
\[ K_j(s) = \frac{1}{2\pi m_j^2} e^{-(s-s_j)^2/(2m_j^2)}, \quad j = r, a, al \quad \text{and} \quad m_j = s_j/8, \quad (2.4) \]
describe the spatial ranges over which the social interactions take place, with \( s_j \),
\( j = r, a, al \) being half the range of the repulsive, attractive and alignment interactions.
Finally, parameters \( m_j \) describe the width of the interaction kernels. Note
that the width of these interaction ranges and interaction kernels depends on the
physiological characteristics of each species (e.g. the interaction ranges for bacteria
are much smaller than the interaction ranges for birds, for example).

\[ u^+ \] \[ u^- \]
\[ x-s \] \[ x \] \[ x+s \]
\[ (a) \]

\[ u^+ \] \[ u^- \]
\[ x-s \] \[ x \] \[ x+s \]
\[ (b) \]

Fig. 1. The movement of a reference individual positioned at \( x \), and its neighbours positioned
at \( x + s \) and \( x - s \). The continuous arrows describe the movement directions of individuals. (a) A
right-moving individual \((u^+); \) (b) A left-moving individual \((u^-)\). The reference individual at \( x \)
will change its direction of movement (with rate \( \lambda^\pm \)) only after perceiving neighbours moving
towards it.
Throughout this paper, we consider a finite domain of length \( L_0 \). For numerical simulations we choose \([0, L_0] = [0, 10]\). At the boundaries, we consider periodic boundary conditions:

\[
u^+(0, t) = \nu^+(L_0, t), \quad \nu^-(L_0, t) = \nu^-(0, t). \tag{2.5}\]

Due to these boundary conditions, we also choose the values of \( m_j \) to ensure that on a finite domain (with wrap-around boundary conditions for the integrals in (2.3)), more than 98\% of the mass of the kernels is inside the domain.\(^24\) Also, \( m_j \) are chosen such that the kernels overlap for less than 2\% of their mass. For this reason (i.e. the separation of spatial ranges for the repulsive/alignment/attractive interactions), we were able to assume that the effect of the social interactions was additive (see Eq. (2.2)). To simplify the analysis, throughout the rest of the paper we ignore the alignment interactions (i.e. \( q_{al} = 0 \)).

**Remark 2.1.** Model (2.1) is the classical prototype of a 1D hyperbolic model that can describe the movement in opposite directions of cells\(^27,28\) or pedestrians.\(^29\) However, the majority of these models are local, incorporating the assumption that cells/individuals can interact only with other cells/individuals at the same position in space. A first nonlocal model, which was the starting point for model (2.1)–(2.3)\(^24\) investigated in this paper, was introduced by Pfistner\(^28\) to study the movement dynamics of myxobacteria. A more detailed discussion on the various types of local and nonlocal hyperbolic models for collective behaviour, and of the various assumptions incorporated in these models, can be found in Eftimie.\(^13\)

**Remark 2.2.** The models for collective behaviour can be classified as: (i) metric-distance models, in which individuals interact with all neighbours within a certain distance, and (ii) topological-distance models, in which individuals interact only with a fixed number of neighbours. Both types of model assumptions are supported by empirical evidence: topological interactions in starling flocks,\(^7\) and metric-distance interactions (given by the interaction zones) in fish shoals\(^30\) or flocking surf scoters.\(^10\) The idea of topological interactions might be connected to the idea of communication mechanisms, by assuming that individuals can interact only with those neighbours that they can perceive. Although experimental evidence supporting this idea is lacking at the moment — due to the difficulty of gathering data regarding group-level communication — it is very plausible that individuals respond only to those neighbours that “catch their attention”. (Partan\(^31\) showed that at behavioural level, the mechanisms that lead to the integration of signals from neighbours involve not only communication but also perception and attention). Ballerini *et al.*\(^7\) do suggest that the number of 6–7 neighbours that were shown to influence the movement of a reference individual in a flock of starlings, is likely the result of cortical incorporation of the visual input. Therefore, model (2.1)–(2.3) combines topological-like interactions (via communication) with metric-distance interactions (via interaction zones).
Finally, the constants $\eta$ and $\theta$ are defined as
\begin{equation}
\eta = \frac{x}{L}, \quad \theta = \frac{y}{L},
\end{equation}
where $x$ and $y$ are the spatial coordinates.

In order to investigate the linear stability of the steady state, we consider perturbations of the form $u(x, t) = u^0(x) + \epsilon u_1(x, t)$, where $u^0(x)$ is the steady state solution and $u_1(x, t)$ is the perturbation. The linearized equation is
\begin{equation}
L(u)(x, t) = \epsilon \left( \frac{\partial u}{\partial t} + \gamma \frac{\partial u}{\partial x} + L_1 - R_2 \right) u^0(x) + \epsilon \left( \frac{\partial u}{\partial x} \right)^2 + \epsilon \left( L_1 - R_2 \right) u_1(x, t),
\end{equation}
where $L_1$ and $R_2$ are defined as
\begin{equation}
L_1 = \lambda_1 + \lambda_2 f(0), \quad R_2 = 2 u^* R_1.
\end{equation}

By solving the eigenvalue problem, we can determine the stability of the steady state. If the real parts of all eigenvalues are negative, the steady state is stable. If there are eigenvalues with positive real parts, the steady state is unstable.

Finally, we consider the effect of spatial heterogeneity on the stability of the steady state. In the presence of spatial heterogeneity, the system becomes spatially inhomogeneous and the linear stability analysis becomes more complicated. However, by considering appropriate symmetries and group actions, we can still determine the stability of the steady state.

The examples considered in this paper are taken from the literature on chemotaxis and pattern formation. The models are based on the assumption of a homogeneous environment, but spatial heterogeneity is often observed in nature. By considering the effect of spatial heterogeneity on the stability of the steady state, we can gain insight into the mechanisms underlying pattern formation and the emergence of complex spatiotemporal structures.
Let us now introduce the matrix operators

$$L_d := \begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} \quad \text{and} \quad L_0 := \begin{pmatrix} -L_1 + R_2 K^- & L_1 - R_2 K^+ \\ L_1 - R_2 K^- & -L_1 + R_2 K^+ \end{pmatrix}. $$

Then, the linear operator $L$ is written as

$$L(u_1) = \begin{pmatrix} \partial_t \\ 0 \end{pmatrix} \begin{pmatrix} u_1^+ \\ u_1^- \end{pmatrix} + L \begin{pmatrix} u_1^+ \\ u_1^- \end{pmatrix}, $$

where $L = -\gamma L_d + L_0$. The $O(2)$-equivariance of system (2.1) implies the $O(2)$-equivariance of the linear operator $L$. Let us write now the linearized system as an abstract differential equation

$$\partial_t u = L(u, \mu), \quad (2.11)$$

with $\mu$ a vector of parameters. As $L_0$ is a bounded operator on $L^p([0, L_0], \mathbb{R}^2)$ (see Theorem 3.4 in Hillen\textsuperscript{12}), it follows that the linear operator $L$ with domain

$$X := \{u = (u^+, u^-) \in W^{1,p}([0, L_0], \mathbb{R}^2) \mid u^\pm(0) = u^\pm(L_0)\},$$

generates a strongly continuous semigroup on $L^p([0, L_0], \mathbb{R}^2)$. We use $X$ as the phase space to (2.11). The symmetries (2.6) act on $X$, and they induce an isotypic decomposition of $X$ described in the following theorem.\textsuperscript{19}

**Theorem 2.1.** For all $n \geq 1$, let $k_n = 2\pi n/L_0$ and consider the subspaces

$$X_n = \{ae^{k_nix} \mid a = (a^+, a^-) \in \mathbb{C}^2 \} \subset X$$

are isomorphic to $\mathbb{C}^2$, $O(2)$-invariant and decompose as $X_n = X_n^1 \oplus X_n^2$, where

$$X_n^1 = \{(v_0e^{k_nix} + \tau_0e^{-k_nix})f_1 \mid v_0 \in \mathbb{C}, \ f_1 = (1, 1)^T\}$$

and

$$X_n^2 = \{(v_1e^{k_nix} + \tau_1e^{-k_nix})f_2 \mid v_1 \in \mathbb{C}, \ f_2 = (-1, 1)^T\}$$

are $O(2)$-irreducible representations of $\mathbb{C}$ dimension 1. The action (2.6) induces the action

$$\theta \cdot (v_0, v_1) = (e^{\kappa n\theta v_0}, e^{\kappa n\theta v_1}),$$

$$\kappa \cdot (v_0, v_1) = (\tau_0, -\tau_1),$$

on $\mathbb{C}^2$.

The $O(2)$ symmetry of $L$ leaves each $X_n$ invariant, and therefore we can define $L_n := L|_{X_n}$. Applying now the convolutions (2.9) to exponentials $e^{k_nix}$ leads to the following Fourier transforms:

$$K^\pm \ast e^{k_nix} = e^{k_nix} \int_{-\infty}^{\infty} K(s)e^{k_nis}ds := \hat{K}^\pm(n)e^{k_nix}. $$
In particular, we have $\hat{K}^-(n) = \hat{K}^+(n)$. In the coordinates given by the decomposition $X_n = X^1_n \oplus X^2_n$, we obtain

$$\mathcal{L}_n \begin{pmatrix} u \\ v \end{pmatrix} e^{k_n i x} = \begin{pmatrix} 0 & -k_n i \gamma \\ R_2(\hat{K}^+(n) - \hat{K}^+(n)) - k_n i \gamma & -2L_1 + R_2(\hat{K}^+(n) + \hat{K}^+(n)) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} e^{k_n i x}. \quad (2.12)$$

The eigenvalues of $L_n$ are

$$\sigma_{n,\pm} = -(L_1 - R_2 \text{Re}(\hat{K}^+(n))) \pm \sqrt{(L_1 - R_2 \text{Re}(\hat{K}^+(n)))^2 - (k_n^2 \gamma^2 - \gamma k_n R_2 \text{Im}(\hat{K}^+(n)))}. \quad (2.13)$$

For $\sigma_n = i\omega$, the nontrivial solution of

$$\begin{pmatrix} i\omega & -k_n i \gamma \\ R_2(\hat{K}^+(n) - \hat{K}^+(n)) - k_n i \gamma & i\omega + (2L_1 - R_2(\hat{K}^+(n) + \hat{K}^+(n))) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0,$$

leads to $\omega u - k_n \gamma v = 0$, and so the eigenspace is $\mathbb{C}\{(k_n \gamma, \omega)^T\}$. The eigenvalues (2.13) are roots of the dispersion relation

$$\sigma^2 + \sigma B(k) + C(k) = 0, \quad (2.14)$$

with

$$B(k) = 2L_1 - R_2 \text{Re}(\hat{K}^+(n)), \quad (2.15a)$$
$$C(k) = \gamma^2 k^2 - \gamma i k R_2 \text{Im}(\hat{K}^+(n)). \quad (2.15b)$$

Figure 2(a) shows the dispersion relation (2.14) for $(q_a, q_r) = (0.63, 3.6)$, when two wave numbers, $k_3$ and $k_4$, become unstable at the same time. Figure 2(b) shows the neutral stability curves ($\sigma(k) = 0$) corresponding to wave numbers $k_3$ (continuous line) and $k_4$ (dashed line). We note that for small $q_{r,a}$ both wave numbers are stable (i.e. $\text{Re}(\sigma(k_{3,4})) < 0$), and for large $q_{r,a}$ both wave numbers are unstable (i.e. $\text{Re}(\sigma(k_{3,4})) > 0$).

### 2.2. Linear modes at bifurcation

Consider the linear system

$$\partial_t u(x, t) = \mathcal{L}_n u(x, t), \quad (2.16)$$

where $u(x, t) = (u^+(x, t), u^-(x, t))^T$ with solutions of the form

$$u(x, t) = \text{Re} \left( \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} e^{k_n i x e^\sigma t} \right) = \text{Re}(v_0 e^{k_n i x e^\sigma t}) f_1 + \text{Re}(v_1 e^{k_n i x e^\sigma t}) f_2.$$
Then, the solutions of the linear system (2.16) can be written as

\[ \Phi^n = k_n \gamma f_1 + \Phi_2 = \omega f_2, \]

where \( f_1 \) and \( f_2 \) are described in Theorem 2.1. Note that

\[ \Phi^n = k_n \gamma + \Phi_2 = \omega + \Phi_2, \]

are:

\[ \Phi^n = k_n \gamma + \Phi_2 = \omega + \Phi_2, \]

Then, the solutions of the linear system (2.16) can be written as

\[ u(x, t) = \text{Re}(w_1 e^{k_n x} e^{i \omega t})(\Phi^n_1 + \Phi_2) + \text{Re}(w_2 e^{k_n x} e^{-i \omega t})(\Phi^n_1 - \Phi_2). \]

The \( O(2) \) group acts on \( \Phi^n_1 \) and \( \Phi_2 \) as follows:

\[ \kappa \cdot (\Phi^n_1 + \Phi_2) = \Phi^n_1 - \Phi_2 \]

and \( \theta \) acts trivially on \( \Phi^n_1 \) and \( \Phi_2 \).

The general real solution at the 3 : 4 H/H bifurcation point is

\[ u(x, t) = w_1 e^{k_n x} e^{i \omega t}(\Phi^3_1 + \Phi_2) + w_2 e^{k_n x} e^{-i \omega t}(\Phi^3_1 - \Phi_2) + \text{Re}(w_3 e^{k_n x} e^{i \omega t}(\Phi^4_1 + \Phi_2) + \text{Re}(w_4 e^{k_n x} e^{-i \omega t}(\Phi^4_1 - \Phi_2)). \]

We rearrange the terms as follows:

\[ u(x, t) = w_1 e^{i \omega t}(e^{k_n x}(\Phi^3_1 + \Phi_2)) + w_2 e^{-i \omega t}(e^{-k_n x}(\Phi^3_1 - \Phi_2)) + w_3 e^{i \omega t}(e^{k_n x}(\Phi^4_1 + \Phi_2)) + w_4 e^{-i \omega t}(e^{-k_n x}(\Phi^4_1 - \Phi_2)) + c.c., \]

where “c.c.” stands for “complex conjugate”. Let

\[ h^i_1 = e^{k_n x}(\Phi^i_1 + \Phi_2) \quad \text{and} \quad h^i_2 = e^{-k_n x}(\Phi^i_1 - \Phi_2). \]
Thus, \( u(x, t) \) is written in terms of the basis \( \{ h_1^n, h_2^n, h_3^n, h_4^n \} \). The action of \( O(2) \) on the elements of the basis is

\[
\kappa \cdot h_1^n = h_2^n, \quad \theta \cdot h_1^n = e^{k n \theta} h_1^n \quad \text{and} \quad \theta \cdot h_2^n = e^{-k n \theta} h_2^n.
\]

This basis is the standard one for the classification of isotropy subgroups in the \( O(2) \) Hopf/Hopf bifurcation.\(^{23}\) The superposition of linear modes at the Hopf/Hopf bifurcation is now given by (in the new coordinates \( z_i \))

\[
u(x, t) = (z_1 h_1^3 + z_2 h_2^3)e^{i \omega_3 t} + (z_3 h_3^4 + z_4 h_4^4)e^{i \omega_4 t} + \text{c.c.} \quad \text{(2.19)}
\]

The action on \( (z_1, z_2, z_3, z_4) \) is

\[
\theta \cdot (z_1, z_2, z_3, z_4) = (e^{k_1 \theta} z_1, e^{-k_3 \theta} z_2, e^{k_4 \theta} z_3, e^{-k_4 \theta} z_4),
\]

\[
\kappa \cdot (z_1, z_2, z_3, z_4) = (z_2, z_1, z_4, z_3). \quad \text{(2.20)}
\]

For the parameter values chosen in this paper (see caption of Fig. 2), the frequencies at the Hopf/Hopf bifurcation point are estimated at \( \omega_3 \approx 0.2461539166 \) and \( \omega_4 \approx 0.3454406196 \). Hence, it is safe to assume that they are not rationally dependent. Therefore a \( T^2 = S^1 \times S^1 \) action generated (after a convenient rescaling) by \( e^{(2\pi i/\omega_3 L_0)\psi_1 \omega_3} \oplus e^{(2\pi i/\omega_4 L_0)\psi_2 \omega_4} \), with \( \psi_1, \psi_2 \in [0, L_0] \), acts on the critical eigenspace \( \mathbb{C}^2 \oplus \mathbb{C}^2 \) as follows:

\[
(\psi_1, \psi_2) \cdot (z_1, z_2, z_3, z_4)
= (e^{2\pi i \psi_1/L_0} z_1, e^{2\pi i \psi_1/L_0} z_2, e^{2\pi i \psi_2/L_0} z_3, e^{2\pi i \psi_2/L_0} z_4). \quad \text{(2.21)}
\]

Together, (2.20) and (2.21) generate the action of \( O(2) \times T^2 \) on the critical eigenspace, and thus on the linear patterns given by \( u(x, t) \). The classification of solutions with respect to their symmetry groups is achieved by computing “isotropy subgroups”, and their general form is obtained by computing “fixed point subspaces”. If we consider the action of a group \( \Gamma \) on a vector space \( V \), then the \textit{isotropy subgroup} of the point \( v \in V \) is

\[
\Gamma_v := \{ \rho \in \Gamma \mid \rho \cdot v = v \}.
\]

For an isotropy subgroup \( \Sigma \subseteq \Gamma \), the \textit{fixed point subspace} of \( \Sigma \) is

\[
\text{Fix}(\Sigma) = \{ v \in V \mid \sigma \cdot v = v, \text{ for all } \sigma \in \Sigma \}.
\]

The importance of the isotropy subgroups and the fixed-point subspaces is in the following flow-invariance property\(^{23}\): \textit{If} \( f \) \textit{is a} \( \Gamma \)-\textit{equivariant (possibly nonlinear) mapping and} \( \Sigma \) \textit{is an isotropy subgroup of} \( \Gamma \), \textit{then} \( f : \text{Fix}(\Sigma) \to \text{Fix}(\Sigma) \). \textit{We will return to this property in Sec. 3.0.3, when we will discuss the equilibrium solutions for the amplitude equations obtained via the weakly nonlinear analysis.}

\textit{Linear patterns at isotropy subgroups.} We now consider the isotropy subgroup classification of possible solutions near the \( O(2) \) Hopf/Hopf bifurcation, and for each solution we describe the linear mode \( u(x, t) \). First, consider the subgroup
$S(k, l, m) = \{(k\psi, \ell\psi, m\psi) \mid \psi \in S^1\}$ of $O(2) \times \mathbb{T}^2$. The action (2.20) and (2.21) has the largest isotropy subgroups for $n = 3, 4$,

$$\tilde{SO}(2) = S(0, 0, 1) \times S(1, n, 0) \quad \text{and} \quad H = Z_2(\kappa) \times Z(L_0/2n, 0, L_0/2),$$

with $H = S(0, 0, 1)$ and $H = S(0, 1, 0)$ (each with two-dimensional fixed point subspaces). The bifurcating solutions in these fixed point subspaces are rotating waves and standing waves, respectively. Figure 3 shows linear patterns for these solutions near the bifurcation point.

Next, we focus on possible tori solutions, and consider the following subgroups of $O(2) \times \mathbb{T}^2$ (see also Chossat et al.\cite{Chossat, Chossat2}):

$$Z(\theta, \psi_1, \psi_2) = \langle \theta, \psi_1, \psi_2 \rangle \subset \tilde{SO}(2) \times \mathbb{T}^2,$$

$$Z_k(\theta, \psi_1, \psi_2) = \langle k\theta, \psi_1, \psi_2 \rangle.$$ These subgroups are found in the four-dimensional fixed point subspaces of the $O(2) \times \mathbb{T}^2$ action. Table 1 shows the isotropy subgroups and the fixed point subspaces defining conditions for these subgroups. Here we follow standard notation for these subgroups, but we keep in mind that $\ell = 3$ and $m = 4$.

![Fig. 3. Rotating wave with (a) $\ell = 3$, (b) $m = 4$ and (c) standing waves.](image)

Table 1. Table of isotropy subgroups with four-dimensional fixed point subspaces of the $O(2) \times \mathbb{T}^2$ action at Hopf/Hopf bifurcation. Recall that $\ell = 3$ and $m = 4$.  

<table>
<thead>
<tr>
<th>Isotropy subgroup</th>
<th>Conditions</th>
<th>$u(x, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(0, 0, 1) \times Z(L_0/2\ell, L_0/2, 0)$</td>
<td>$z_3 = z_4 = 0$</td>
<td>$(z_1h_1^4 + z_2h_2^4)e^{i\omega t} + \text{c.c.}$</td>
</tr>
<tr>
<td>$S(0, 1, 0) \times Z(L_0/2m, 0, L_0/2)$</td>
<td>$z_1 = z_2 = 0$</td>
<td>$(z_3h_1^4 + z_4h_2^4)e^{i\omega t} + \text{c.c.}$</td>
</tr>
<tr>
<td>$S(1, \ell, m)$</td>
<td>$z_1 = z_3 = 0$</td>
<td>$z_2e^{i\omega t}h_1^2 + z_4e^{i\omega t}h_2^2 + \text{c.c.}$</td>
</tr>
<tr>
<td>$S(1, \ell, -m)$</td>
<td>$z_1 = z_4 = 0$</td>
<td>$z_2e^{i\omega t}h_1^2 + z_4e^{i\omega t}h_2^2 + \text{c.c.}$</td>
</tr>
<tr>
<td>$Z_2^0 \times Z(L_0/2, \ell L_0/2, mL_0/2)$</td>
<td>$z_1 = z_2, z_3 = z_4$</td>
<td>$z_1\Omega_3 + z_4\Omega_4 + \text{c.c.}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Omega_i := e^{i\omega t}(h_1^i + h_2^i)$, $i = 3, 4$.</td>
</tr>
<tr>
<td>$Z_2(0, 0, L_0/2) \times Z(L_0/2, \ell L_0/2, mL_0/2)$</td>
<td>$z_1 = z_2, z_3 = -z_4$</td>
<td>$z_3\Omega_3 + z_4e^{i\Omega t}(h_1^4 - h_2^4) + \text{c.c.}$</td>
</tr>
</tbody>
</table>
Table 2. Table of isotropy subgroups with four-dimensional fixed point subspaces of the $O(2) \times T^2$ action, near the Hopf/Hopf bifurcation point. Recall that $\ell = 3$ and $m = 4$.

<table>
<thead>
<tr>
<th>Isotropy subgroup</th>
<th>$u(x, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S(0, 0, 1) \times Z(L_0/2\ell, L_0/2, 0)$</td>
<td>$2(k_1) e^{i\omega_1 t}(z_1 e^{ik_3 x} + z_2 e^{-ik_3 x}) + c.c.$</td>
</tr>
<tr>
<td>$S(0, 1, 0) \times Z(L_0/2m, 0, L_0/2)$</td>
<td>$2(k_4) e^{i\omega_4 t}(z_3 e^{ik_4 x} + z_4 e^{-ik_4 x}) + c.c.$</td>
</tr>
<tr>
<td>$S(1, \ell, m)$</td>
<td>$2\gamma[k_3 z_2 e^{ik_{3\ell} x} e^{i\omega_3 t} + k_4 z_4 e^{-ik_{4\ell} x} e^{i\omega_4 t}] + c.c.$</td>
</tr>
<tr>
<td>$Z_2 \times Z(L_0/2, \ell L_0/2, m L_0/2)$</td>
<td>$z_1 e^{i\omega_3 t} \cos(k_3 x) + z_3 e^{i\omega_4 t} \cos(k_4 x) + c.c.$</td>
</tr>
<tr>
<td>$Z_2 \times Z(L_0/2, \ell L_0/2, m L_0/2)$</td>
<td>$z_1 e^{i\omega_3 t} \cos(k_3 x) + z_3 e^{i\omega_4 t} 4 \cos(k_4 x) + c.c.$</td>
</tr>
</tbody>
</table>

Fig. 4. Total density $u(x, t)$ for submaximal isotropy subgroup solutions: (a) $S(0, 0, 1)$, (b) $S(1, \ell, m)$, (c) $Z_2 \times Z(L_0/2, \ell L_0/2, m L_0/2)$ and (d) $Z_2 \times Z(L_0/2, \ell L_0/2, m L_0/2)$.

Table 2 shows $u(x, t) = u^+(x, t) + u^-(x, t)$ for each isotropy subgroup (note that there is no contribution to $u(x, t)$ from the $\Phi_2$ terms). Finally, Fig. 4 shows the contourplots of $u(x, t)$ as given by the eigenfunctions, for the isotropy subgroups $S(0, 0, 1)$, $S(1, \ell, m)$, $Z_2 \times Z(L_0/2, \ell L_0/2, m L_0/2)$ and $Z_2 \times Z(L_0/2, \ell L_0/2, m L_0/2)$. 


Nonlinear Analysis of $H/H$ Bifurcations

Fig. 5. Spatio-temporal patterns observed near the 3:4 $H/H$ bifurcation point. Panels (a)–(d) show the total density $u = u^+ + u^-$. Panels (a$'$)–(c$'$) show the density of right-moving individuals $u^+$. Panels (a$''$)–(c$''$), (d$'$) show the density of left-moving individuals. (a)–(a$'$) Standing waves (SW), corresponding to $k_d$, for $q_a = 0.72, q_r = 3.55$; (b)–(b$'$) modulated standing waves (MSW), for $q_a = 0.66, q_r = 3.56$; (c)–(c$'$) rotating waves (RW), for $q_a = 0.72, q_r = 3.55$; (d), (d$'$) modulated rotating waves (MRW), for $q_a = 0.76, q_r = 3.56$; (e) spatially homogeneous solutions (SH), for $q_{r,a}$ in the parameter region where $\text{Re}(\sigma(k_{3,4})) < 0$: $q_a = 0.51, q_r = 3.53$. The rest of the parameters are: $q_{c,t} = 0, \lambda_1 = 0.2, \lambda_2 = 0.9, \gamma = 0.1, L_0 = 10$. 
2.3. Numerical simulations

Figure 5 shows the patterns (and the absence of spatio-temporal patterns) observed near the H/H bifurcation point identified in Fig. 2. For numerical simulations, we used a simple, first-order up-wind scheme that incorporates periodic boundary conditions. The simulations were also checked with a second-order McCormack scheme. The integrals were discretised using the Simpson’s rule. The initial conditions for the simulations are small random perturbations of spatially homogeneous steady states \((u^+, u^-) = (u^*, u^*)\), with \(u^* = 1.0\).

The numerical results in Fig. 5 show four types of spatio-temporal patterns that can emerge around the H/H point: standing waves corresponding to one wave number, here \(k_4\) (panels (a), (a'), (a'')), modulated standing waves with isotropy subgroups \(\mathbb{Z}_2(k) \times Z(L_0/2, \ell L_0/2, m L_0/2)\) or \(\mathbb{Z}_6(0, 0, L_0/2) \times Z(L_0/2, \ell L_0/2, m L_0/2)\), with \(\ell = 3, m = 4\) (panels (b), (b'), (b'')); similar to Fig. 4(c), (d)), rotating waves (panels (c), (c'), (c'')) and modulated rotating waves (panels (d), (d')). The small initial perturbations can also decrease in amplitude, and in this case the dynamics approaches a spatially homogeneous steady state (panel (e)). Note that for rotating waves and modulated rotating waves all populations move in the same direction (i.e. the \(u^-\) populations, which are small here, move in the direction of the larger population, the \(u^+\) population here). Also, these two waves seem to occur further away from the H/H point (i.e. rotating waves for \(q_a \geq 0.72\) and modulated rotating waves for \(q_a \geq 0.75\)).

Multiple simulations with various parameter values show that near the H/H point, the spatio-temporal patterns are usually unstable, with the dynamics approaching the spatially homogeneous states (e) (either the initial \((u^*, u^*)\) state or the state \((u^+_*, u^-_*)\) mentioned in the previous section). These results raise the question of whether it is possible to have stable spatio-temporal patterns (at least in some restricted parameter space near the H/H point). In the next section, we start the analytical investigation of these patterns. We identify the amplitude curves for all bifurcating solutions that could possibly emerge near this 3:4 H/H point, and discuss the stability of these curves.

3. Weakly Nonlinear Analysis

In the following, we investigate the types of patterns that can arise around the 3:4 Hopf/Hopf bifurcation point identified previously, as we vary two parameters, \(q_a\) and \(q_r\) around their critical values \(q_a^0 = 0.637\) and \(q_r^0 = 3.607\). At this bifurcation point, the imaginary eigenvalues of the dispersion relation are \(\sigma(k_3; q_a^0, q_r^0) = \pm i \omega(k_3)\) and \(\sigma(k_4; q_a^0, q_r^0) = \pm i \omega(k_4)\). As shown in Sec. 2.2, at the linear level, the solution can be expressed as

\[
\begin{align*}
  u^{\pm}(x, t) &\propto e^{i \omega(k_3) t + i k_3 x} + e^{i \omega(k_3) t - i k_3 x} + e^{i \omega(k_4) t + i k_4 x} + e^{i \omega(k_4) t - i k_4 x} + \text{c.c.} \\
  &\propto e^{i \nu_r \epsilon^2 (k_3 - k_4) t + i (k_3 - k_4) x}.
\end{align*}
\]  

(3.1)

In the neighbourhood of the critical point \((q_a^0, q_r^0)\), we have

\[
q_r = q_r^0 + \nu_r \epsilon^2, \quad q_a = q_a^0 + \nu_a \epsilon^2,
\]

(3.2)
where \( \nu_{r,a} = \pm 1 \) and \( 0 < \epsilon \ll 1 \). The dispersion relation can now be written as a power series about the critical point, as follows:

\[
\sigma(k_n; q_a, q_r) = \sigma(k_n; q_a^0, q_r^0) + \frac{\partial \sigma}{\partial q_a} \epsilon^2 \nu_a + \frac{\partial \sigma}{\partial q_r} \epsilon^2 \nu_r. \tag{3.3}
\]

Thus, we can re-write the exponential terms that appear in the solution (3.1) as

\[
e^{\sigma(k_n; q_a, q_r) t \pm ik_n x} \approx \alpha(t^2; \nu_a, \nu_r) e^{\sigma(k_n; q_a^0, q_r^0) t \pm ik_n x}. \tag{3.4}
\]

Therefore, the solution evolves on two different time scales: a fast time scale \( t^* = t \) (in the exponential term) and a slow time scale \( T = \epsilon^2 t \) (in the amplitude term). In the limit \( \epsilon \to 0 \), we treat these two scales as being independent.\(^{33}\) For simplicity, we can drop the asterisk from the fast time scale. If we expand the solution \( u^\pm(x, t, \epsilon, T) \) of (2.1) in powers of \( \epsilon \)

\[
u^+(x, t, \epsilon, T) = u^* + \epsilon u^+_1 + \epsilon^2 u^+_2 + \epsilon^3 u^+_3 + O(\epsilon^4) \tag{3.5}
\]

and then substitute it back into the nonlinear system (2.1). This system can then be written as

\[
0 = \mathbf{N} \left( \sum_{j \geq 1} \epsilon^j \mathbf{u}_j \right) \approx \sum_{j \geq 1} \mathbf{N}_j(\mathbf{u}_j) = \sum_{j \geq 1} L(\mathbf{u}_j) - N_j - E_j. \tag{3.6}
\]

Here \( \mathbf{u}_j = (u^+_j, u^-_j)^T \), \( L \) is the linear operator defined by (2.8), \( N_j \) is the nonlinear operator (at \( O(\epsilon^j) \)) which contains the terms \( u^+_j, u^-_j \), etc. and \( E_j \) is the nonlinear operator (at \( O(\epsilon^j) \)) which contains the slow-time derivatives (\( \partial u^\pm / \partial T \)) and the terms multiplied by \( \nu_a, \nu_r \).

### 3.0.1. Function spaces

As we will see shortly, at \( O(\epsilon^1) \), equation \( L(\mathbf{u}_1) = 0 \) has a nontrivial solution. Therefore, at each \( O(\epsilon^j) \), \( j \geq 2 \), Eq. (3.6) will have solutions if and only if \( N_j + E_j \) satisfy the Fredholm alternative.\(^{34}\) To investigate this aspect, we need to ensure that the linear operator \( L \) is compact. We consider the following Hilbert space:

\[
Y = \left\{ \mathbf{u}(x, \tau) \mid (x, t) \in [0, L_0] \times [0, \infty), \text{s.t. } \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_0^{L_0} |\mathbf{u}| dx d\tau < \infty \right\}, \tag{3.7}
\]

with the \( O(2) \)-invariant inner product

\[
\langle \mathbf{u}, \mathbf{w} \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_0^{L_0} \frac{r_n}{n} (u^+ \dot{w}^- + u^- \dot{w}^+) dx d\tau, \tag{3.8}
\]

with \( \mathbf{u} = (u^+, u^-)^T, \mathbf{w} = (w^+, w^-)^T \). One can easily check that for all \( \theta \in \text{SO}(2), \quad \langle \theta \cdot \mathbf{u}, \theta \cdot \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle \) and \( \langle \kappa \cdot \mathbf{u}, \kappa \cdot \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle \).
Consider now the space

\[ V_{bc} = \{ u = (u^+, u^-) \in Y \mid (\partial_x u^+, \partial_x u^-) \in Y, \text{ and } u^\pm \text{ satisfy } (2.5) \}, \tag{3.9} \]

endowed with the norm

\[ \| u \|_Y^2 = \| (u^+, u^-) \|_Y^2 + \| (\partial_x u^+, \partial_x u^-) \|_Y^2. \tag{3.10} \]

Here, \(| \cdot |_Y\) is the norm associated with the inner product (3.8). Note that since \(u^\pm\) are bounded on \(L^\infty([0, L_0] \times [0, T])\),\(^{17}\) and \(L^1([0, L_0] \times [0, T])\),\(^{35}\) and \(\partial_x u^\pm\) are bounded on \(L^1([0, L_0] \times [0, T])\),\(^{35}\) it can be shown that \(\partial_t u^\pm\) are bounded on \(L^1([0, L_0] \times [0, T])\). Therefore, using the relation between the \(L^1\) and \(L^2\) norms, we obtain that the limits \(\lim_{T \to \infty} \frac{1}{T}\| (u^+, u^-) \|_{L^2([0, L_0] \times [0, T])}^2\) and \(\lim_{T \to \infty} \frac{1}{T}\| (\partial_t u^+ + \gamma \partial_x u^+, \partial_t u^- - \gamma \partial_x u^-) \|_{L^2([0, L_0] \times [0, T])}^2\) are finite. Thus the linear operator \(L: V_{bc} \to Y\) given by (2.8) is bounded. Following the approach in Mallet-Paret\(^{36}\) or Knit and Recke\(^{37}\) it can be shown that the operator \(L: V_{bc} \to Y\) is a Fredholm operator.

### 3.0.2. \(O(\epsilon^1)\) solutions

We start the weakly nonlinear analysis of the steady state \((u^*, u^*)\) near the 3:4 codimension-2 bifurcation point \((q_d^0, q_d^0)\) by identifying the coefficients of the solution at each \(O(\epsilon^1)\). At \(O(\epsilon^1)\) we write

\[
\begin{bmatrix}
  u_1^+ \\
  u_1^-
\end{bmatrix} = \begin{pmatrix}
  \alpha_1(T) & \alpha_2(T) \\
  \beta_1(T) & \beta_2(T)
\end{pmatrix} \begin{bmatrix}
  v_{11} \\
  v_{12} \\
  v_{21} \\
  v_{22}
\end{bmatrix} e^{i\omega_1 t - ik_3 x} + \frac{1}{\epsilon} \begin{pmatrix}
  w_{11} \\
  w_{12} \\
  w_{21} \\
  w_{22}
\end{bmatrix} e^{i\omega_1 t - ik_4 x} + \text{c.c.} \tag{3.11}
\]

Since the nonlinear operators are \(N_1 = 0, E_1 = 0\), we calculate the coefficient vectors \((v_{11}, v_{12})^T, (v_{21}, v_{22})^T, (w_{11}, w_{12})^T\) and \((w_{21}, w_{22})^T\) that appear in Eq. (3.11) by solving

\[
L(u_1) = 0, \tag{3.12}
\]

with \(u_1 = (u_1^+, u_1^-)\). Note that applying the linear operator \(L\) (2.8) to solutions \(u_j^\pm \propto e^{i\omega(k_n)\pm ik_n x}\), leads to

\[
L_{\omega_n, \pm k_n} = \begin{pmatrix}
  i\omega_n \pm \gamma k_n + L_1 - R_2 K^\pm(k_n) & -L_1 + R_2 K^\pm(k_n) \\
  -L_1 + R_2 K^\pm(k_n) & i\omega_n \pm \gamma k_n + L_1 - R_2 K^\pm(k_n)
\end{pmatrix}. \tag{3.13}
\]

To simplify the notation, we defined here \(\omega_n = \omega(k_n)\). We use this linear operator to calculate the previous coefficient vectors (shown in Appendix A), which correspond to the expressions obtained in (2.17).
At $O(\epsilon^2)$, $N_2 \neq 0$ and $E_2 = 0$. The nonlinear equation

$$L \begin{pmatrix} u_2^+ \\ u_2^- \end{pmatrix} = N_2$$

has a nontrivial solution iff $N_2 + E_2$ satisfies the Fredholm alternative. Therefore $N_2 + E_2$ has to be orthogonal to the bounded solution of the adjoint homogeneous problem $L^\top(\mathbf{u}) = 0$. The adjoint operator is defined by

$$L^\top_{\omega_n, \pm k_n} = \begin{pmatrix} -i\omega_n \mp \gamma ik_n + L_1 - R_2 \hat{K}^\top(k_n) & -L_1 + R_2 \hat{K}^\top(k_n) \\ -L_1 + R_2 \hat{K}^\top(k_n) & -i\omega_n \pm \gamma ik_n + L_1 - R_2 \hat{K}^\top(k_n) \end{pmatrix}.$$  

The solution of the adjoint homogeneous problem is

$$\mathbf{u} = a_1^*(T) \mathbf{v}_1^* e^{i(\omega t - ik_n x)} + a_2^*(T) \mathbf{v}_2^* e^{i(\omega t + ik_n x)} + \beta^*(T) \mathbf{w}_1^* e^{i(\omega t - ik_n x)} + \beta^*(T) \mathbf{w}_2^* e^{i(\omega t + ik_n x)} + c.c.$$ 

The coefficients $\mathbf{v}_1^*$ and $\mathbf{w}_1^*$ are given in Appendix A. Simple calculations show that $N_2 + E_2$ satisfies the Fredholm alternative (that is, $(\mathbf{u}, N_2 + E_2) = 0$).

By substituting the expressions for $u_2^+$ into the nonlinear terms of $N_2 + E_2$, we can calculate the solution at $O(\epsilon^2)$ (see also Appendix A):

$$\begin{pmatrix} u_2^+ \\ u_2^- \end{pmatrix} = \alpha_3(T) \mathbf{v}_3 e^{i(\omega t + ik_n x)} + \alpha_4(T) \mathbf{v}_4 e^{i(\omega t - ik_n x)} + \beta_3(T) \mathbf{w}_3 e^{i(\omega t + ik_n x)} + \beta_4(T) \mathbf{w}_4 e^{i(\omega t - ik_n x)} + G^1|\alpha_1|^2 + G^2|\alpha_2|^2 + G^3|\beta_1|^2 + G^4|\beta_2|^2 + G^6 \alpha_3 \beta_3 e^{2i(\omega t - ik_n x)} + G^7 \alpha_4 \beta_4 e^{2i(\omega t + ik_n x)} + G^8 \alpha_5 \beta_5 e^{i(\omega t - ik_n x)} + G^9 \alpha_6 \beta_6 e^{i(\omega t + ik_n x)} + G^{10} \alpha_7 \beta_7 e^{i(\omega t - ik_n x)} + G^{11} \alpha_8 \beta_8 e^{i(\omega t + ik_n x)} + G^{12} \beta_9 e^{i(\omega t - ik_n x)} + G^{13} \alpha_9 \beta_9 e^{i(\omega t + ik_n x)} + G^{14} \alpha_1 \beta_1 e^{i(\omega t + ik_n x)} + G^{15} \alpha_2 \beta_2 e^{i(\omega t - ik_n x)} + G^{16} \alpha_3 \beta_3 e^{i(\omega t + ik_n x)} + G^{17} \alpha_4 \beta_4 e^{i(\omega t - ik_n x)} + G^{18} \alpha_5 \beta_5 e^{i(\omega t + ik_n x)} + G^{19} \alpha_6 \beta_6 e^{i(\omega t - ik_n x)} + G^{20} \alpha_7 \beta_7 e^{i(\omega t + ik_n x)} + c.c.$$  

The coefficient vectors $G^j, j = 1, \ldots, 20$, which are shown in Appendix A, appear from the nonlinear operator $N_2$.

Finally, at $O(\epsilon^3)$, the nonlinear operators are nonzero ($N_3, E_3 \neq 0$; see Appendix A). For the Fredholm alternative, we are interested only in the terms $e^{i(\omega t + ik_n x)}$ that appear in $N_3 + E_3$ (since all other terms are orthogonal to the solution $\mathbf{u} \propto e^{i(\omega t - ik_n x)}$ of the adjoint homogeneous problem). Imposing the orthogonality condition $(\mathbf{u}, N_3 + E_3) = 0$ leads to the following four complex amplitude
The coefficients equations:

\[
\frac{d\alpha_1(T)}{dT} = -X_1\alpha_1 + X_2\alpha_1|\alpha_1|^2 + X_3\alpha_1|\alpha_2|^2 + X_4\alpha_1|\beta_1|^2 + X_5\alpha_1|\beta_2|^2, \quad (3.17a)
\]

\[
\frac{d\alpha_2(T)}{dT} = -Y_1\alpha_2 + Y_2\alpha_2|\alpha_1|^2 + Y_3\alpha_2|\alpha_2|^2 + Y_4\alpha_2|\beta_1|^2 + Y_5\alpha_2|\beta_2|^2, \quad (3.17b)
\]

\[
\frac{d\beta_1(T)}{dT} = -Z_1\beta_1 + Z_2\beta_1|\alpha_1|^2 + Z_3\beta_1|\alpha_2|^2 + Z_4\beta_1|\beta_1|^2 + Z_5\beta_1|\beta_2|^2, \quad (3.17c)
\]

\[
\frac{d\beta_2(T)}{dT} = -\Psi_1\beta_2 + \Psi_2\beta_2|\alpha_1|^2 + \Psi_3\beta_2|\alpha_2|^2 + \Psi_4\beta_2|\beta_1|^2 + \Psi_5\beta_2|\beta_2|^2. \quad (3.17d)
\]

The coefficients \(X_m, Y_m, Z_m\) and \(\Psi_m, m = 1, 2, 3, 4, 5\), are given in Appendix A. Note that, for the parameter values used in this paper (see the caption of Fig. 5), the calculations show the following important equalities:

\[
X_1 = Y_1, \quad Z_1 = \Psi_1, \quad (3.18a)
\]

\[
X_2 + X_3 = Y_2 + Y_3 = 2.024, \quad Z_4 + Z_5 = \Psi_4 + \Psi_5 = 3.781, \quad (3.18b)
\]

\[
X_4 + X_5 = Y_4 + Y_5 \approx 4.1, \quad Z_2 + Z_3 = \Psi_2 + \Psi_3 \approx 2.2. \quad (3.18c)
\]

We will return to these equalities shortly, when we will discuss the types of bifurcating solutions.

By rewriting

\[
\alpha_1(T) = \Lambda_1(T)e^{i\Theta_1(T)}, \quad \alpha_2(T) = \Lambda_2(T)e^{i\Theta_2(T)},
\]

\[
\beta_1(T) = \Lambda_3(T)e^{i\Theta_3(T)}, \quad \beta_2(T) = \Lambda_4(T)e^{i\Theta_4(T)},
\]

we obtain four coupled equations for real amplitudes \((\Lambda_j, j = 1, 2, 3, 4)\) and four coupled phase equations \((\Theta_j, j = 1, 2, 3, 4)\). Here, we ignore the phase equations and focus only on the real amplitudes.

\[
\frac{d\Lambda_1(T)}{dT} = \Lambda_1(T)(-\text{Re}(X_1) + \Lambda_1(T)^2 \text{Re}(X_2) + \Lambda_2(T)^2 \text{Re}(X_3))
\]

\[
\quad + \Lambda_3(T)^2 \text{Re}(X_4) + \Lambda_4(T)^2 \text{Re}(X_5)), \quad (3.19a)
\]

\[
\frac{d\Lambda_2(T)}{dT} = \Lambda_2(T)(-\text{Re}(Y_1) + \Lambda_1(T)^2 \text{Re}(Y_2) + \Lambda_2(T)^2 \text{Re}(Y_3))
\]

\[
\quad + \Lambda_3(T)^2 \text{Re}(Y_4) + \Lambda_4(T)^2 \text{Re}(Y_5)), \quad (3.19b)
\]

\[
\frac{d\Lambda_3(T)}{dT} = \Lambda_3(T)(-\text{Re}(Z_1) + \Lambda_1(T)^2 \text{Re}(Z_2) + \Lambda_2(T)^2 \text{Re}(Z_3))
\]

\[
\quad + \Lambda_3(T)^2 \text{Re}(Z_4) + \Lambda_4(T)^2 \text{Re}(Z_5)), \quad (3.19c)
\]

\[
\frac{d\Lambda_4(T)}{dT} = \Lambda_4(T)(-\text{Re}(\Psi_1) + \Lambda_1(T)^2 \text{Re}(\Psi_2) + \Lambda_2(T)^2 \text{Re}(\Psi_3))
\]

\[
\quad + \Lambda_3(T)^2 \text{Re}(\Psi_4) + \Lambda_4(T)^2 \text{Re}(\Psi_5)). \quad (3.19d)
\]

Consider now the group \(\mathbb{Z}_2^4\) generated by elements of the form \((\delta_1, \delta_2, \delta_3, \delta_4)\), where \(\delta_i = \pm 1\) for \(i = 1, 2, 3, 4\), and \(\mathbb{Z}_2^4\) acts on \(\mathbb{R}^4 = \{(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)\}\)
componentwise. Then, the system of equations (3.19) is $\mathbb{Z}_2^4$-equivariant. Moreover, let us restrict ourselves to the fixed-point subspaces

$$\text{Fix}(Z_2(\delta_3) \times Z_2(\delta_4)) = \{(\Lambda_1, \Lambda_2, 0, 0)\} \quad \text{and}$$

$$\text{Fix}(Z_2(\delta_1) \times Z_2(\delta_2)) = \{(0, 0, \Lambda_3, \Lambda_4)\}$$

(3.20)

and consider the operation $\kappa \cdot (\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4) = (\Lambda_2, \Lambda_1, \Lambda_4, \Lambda_3)$ acting on these subspaces. Note that $\kappa$ is not a symmetry of (3.19). Then, due to the equalities in (3.18), $\text{Fix}(\kappa) \cap \text{Fix}(Z_2(\delta_3) \times Z_2(\delta_4))$ and $\text{Fix}(\kappa) \cap \text{Fix}(Z_2(\delta_1) \times Z_2(\delta_2))$ are one-dimensional flow-invariant subspaces of (3.19). Because of equality (3.18c), $\text{Fix}(\kappa)$ is also a flow-invariant subspace. Indeed, setting $\Lambda_1 = \Lambda_2$ and $\Lambda_3 = \Lambda_4$, (3.19) becomes

$$\frac{d\Lambda_1(T)}{dT} = \Lambda_1(T)(-\text{Re}(X_1) + \Lambda_1(T)^2(\text{Re}(X_2 + X_3)) + \Lambda_3(T)^2(\text{Re}(X_4 + X_5))),$$

$$\frac{d\Lambda_1(T)}{dT} = \Lambda_1(T)(-\text{Re}(X_1) + \Lambda_1(T)^2(\text{Re}(Y_2 + Y_3)) + \Lambda_3(T)^2(\text{Re}(Y_4 + Y_5))),$$

$$\frac{d\Lambda_3(T)}{dT} = \Lambda_3(T)(-\text{Re}(Z_1) + \Lambda_1(T)^2(\text{Re}(Z_2 + Z_3)) + \Lambda_3(T)^2(\text{Re}(Z_4 + Z_5))),$$

$$\frac{d\Lambda_3(T)}{dT} = \Lambda_3(T)(-\text{Re}(Z_1) + \Lambda_1(T)^2(\text{Re}(\Psi_2 + \Psi_3)) + \Lambda_3(T)^2(\text{Re}(\Psi_4 + \Psi_5))).$$

Thus, any solution with initial condition in $\text{Fix}(\kappa)$ stays in $\text{Fix}(\kappa)$.

3.0.3. Equilibrium solutions for the amplitude equations

In the following, we use a symmetry-based classification of the equilibrium solutions of (3.19), to obtain equations for the amplitudes of the bifurcating solutions. To this end, we will link these equilibrium solutions to solutions (3.11) with isotropy subgroups of the $O(2) \times \mathbb{T}^2$ action (2.20).

We begin with the symmetry-based classification of equilibrium solutions of (3.19). The isotropy subgroups of the $\mathbb{Z}_2^4$-action are the various combinations of $\delta_i = \pm 1$, $i = 1, 2, 3, 4$. Thus, the fixed-point subspaces of isotropy subgroups in $\mathbb{R}^4$ are determined by which of the coordinates $\Lambda_1, \ldots, \Lambda_4$ are equal to zero (see for instance (3.20)). Recall that fixed-point subspaces of the $\mathbb{Z}_2^4$ action are flow-invariant for (3.19).

<table>
<thead>
<tr>
<th>Table 3. Labelling of isotropy subgroups of the $O(2) \times \mathbb{T}^2$ action.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma_1 = S(0, 0, 1) \times S(1, 3, 0)$</td>
</tr>
<tr>
<td>$\Sigma_3 = S(0, 0, 1) \times Z_2(\alpha) \times Z\left(\frac{4\pi}{3}, \frac{4\pi}{3}, 0\right)$</td>
</tr>
<tr>
<td>$\Sigma_5 = S(0, 0, 1) \times Z\left(\frac{4\pi}{3}, \frac{4\pi}{3}, 0\right)$</td>
</tr>
<tr>
<td>$\Sigma_7 = S(1, 3, 4)$</td>
</tr>
<tr>
<td>$\Sigma_9 = Z_2(\alpha) \times Z\left(\frac{4\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3}\right)$</td>
</tr>
</tbody>
</table>


Table 4. Solutions $u_i^+$ obtained from the equilibrium solutions of the amplitude equations. The isotropy subgroups are listed in Table 3. SH: spatially homogeneous, RW: rotating wave, SW: standing wave, MSW: modulated standing wave, MRW: modulated rotating wave. Note that for 9 and 10, the + sign is for $\Sigma_9$ and the minus for $\Sigma_{10}$.

<table>
<thead>
<tr>
<th>#</th>
<th>Solution</th>
<th>Isot. sub.</th>
<th>Defining conditions</th>
<th>Weakly-nonlinear solution $u_1(x, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>SH</td>
<td>$O(2) \times T^2$</td>
<td>$\Lambda_1 = 0$, $i = 1, 2, 3, 4$.</td>
<td>$0$</td>
</tr>
<tr>
<td>1</td>
<td>RW</td>
<td>$\Sigma_1$</td>
<td>$\Lambda_1^2 = \frac{\Lambda_2}{\Lambda_3}$, $\Lambda_2 = \Lambda_3 = \Lambda_4 = 0$.</td>
<td>$\alpha_1(T) v_1 e^{i \omega_0 t + k_3 x}$</td>
</tr>
<tr>
<td>2</td>
<td>RW</td>
<td>$\Sigma_2$</td>
<td>$\Lambda_3^2 = \frac{\Lambda_3}{\Lambda_2}$, $\Lambda_1 = \Lambda_2 = \Lambda_3 = 0$.</td>
<td>$\beta_2(T) w_2 e^{i \omega_0 t + k_4 x}$</td>
</tr>
<tr>
<td>3</td>
<td>SW</td>
<td>$\Sigma_3$</td>
<td>$\Lambda_1^2 = \Lambda_2^2 = \frac{\text{Re}(X_1)}{\text{Re}(X_2) + \text{Re}(X_3)}$, $\Lambda_3 = \Lambda_4 = 0$.</td>
<td>$\alpha_1(T) (v_1 e^{i \omega_0 t + k_3 x} + v_2 e^{i \omega_0 t - k_3 x})$</td>
</tr>
<tr>
<td>4</td>
<td>SW</td>
<td>$\Sigma_4$</td>
<td>$\Lambda_1^2 = \Lambda_3^2 = \frac{\text{Re}(Z_1)}{\text{Re}(Z_4) + \text{Re}(Z_5)}$, $\Lambda_1 = \Lambda_2 = 0$.</td>
<td>$\beta_1(T) (w_1 e^{i \omega_0 t + k_4 x} + w_2 e^{i \omega_0 t - k_4 x})$</td>
</tr>
<tr>
<td>5</td>
<td>MSW</td>
<td>$\Sigma_5$</td>
<td>$\Lambda_1 = \Lambda_4 = 0$, $\Lambda_1^2 \neq \Lambda_3^2$,</td>
<td>$\alpha_1(T) v_1 e^{i \omega_0 t + k_3 x} + \alpha_2(T) v_2 e^{i \omega_0 t - k_3 x}$</td>
</tr>
<tr>
<td>6</td>
<td>MSW</td>
<td>$\Sigma_6$</td>
<td>$\Lambda_1 = \Lambda_2 = 0$, $\Lambda_1^2 \neq \Lambda_3^2$,</td>
<td>$\beta_1(T) w_1 e^{i \omega_0 t + k_4 x} + \beta_2(T) w_2 e^{i \omega_0 t - k_4 x}$</td>
</tr>
<tr>
<td>7</td>
<td>MRW</td>
<td>$\Sigma_7$</td>
<td>$\Lambda_1 = \Lambda_3 = 0$,</td>
<td>$\alpha_2(T) v_2 e^{i \omega_0 t - k_3 x} + \beta_2(T) w_2 e^{i \omega_0 t - k_4 x}$</td>
</tr>
<tr>
<td>8</td>
<td>MRW</td>
<td>$\Sigma_8$</td>
<td>$\Lambda_1 = \Lambda_4 = 0$,</td>
<td>$\alpha_2(T) v_2 e^{i \omega_0 t - k_3 x} + \beta_1(T) w_1 e^{i \omega_0 t + k_4 x}$</td>
</tr>
<tr>
<td>9, 10</td>
<td>MRW</td>
<td>$\Sigma_{9, 10}$</td>
<td>$\Lambda_1 = \Lambda_2$, $\Lambda_3 = \pm \Lambda_4$,</td>
<td>$\alpha_1(T) (v_1 e^{i \omega_0 t + k_3 x} + v_2 e^{i \omega_0 t - k_3 x})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\frac{\text{Re}(X_2 + X_3)}{\text{Re}(X_2 + X_5)}$, $\Lambda_1^2 = \frac{\text{Re}(X_1)}{\text{Re}(X_1)}$,</td>
<td>$+ \beta_1(T) (w_1 e^{i \omega_0 t + k_4 x} \pm w_2 e^{i \omega_0 t - k_4 x})$</td>
</tr>
</tbody>
</table>
In Table 4, we list the (conjugacy classes of) isotropy subgroups of the $O(2) \times \mathbb{T}^2$ action (third column; see also Table 3), the defining conditions for the equilibria of (3.19) (fourth column) and the form of the solutions (3.11) (fifth column). Recall that two subgroups $H_1$ and $H_2$ of a group $G$ are conjugate in $G$ if there exists an element $g \in G$ such that $g^{-1}H_1 g = H_2$. As a consequence, if $x_1 \in \text{Fix}(H_1)$, then $g^{-1}x_1 \in \text{Fix}(H_2)$. Therefore, the existence of a solution for an isotropy subgroup $H_1$ automatically implies a solution with a conjugate isotropy subgroup $H_2$. For this reason, we do not list in Table 4 all possible solutions (RW, SW, etc.) of the full system (2.1).

The case with only one $\Lambda_i \neq 0$, say $\Lambda_1 \neq 0$, yields $\Lambda_1^2 = \text{Re}(X_1)/\text{Re}(X_2)$ (see solution #1 in Table 4). The other cases with only one nonzero $\Lambda_i$ are similar (see solution #2 in Table 4). For exactly two nonzero amplitudes $\Lambda_{i,j} \neq 0$ (e.g. $\Lambda_1 = \Lambda_2 \neq 0$ as for solution #3 in Table 4), we distinguish the cases of (3.20) (i.e. solutions #3 and #4) from the other cases because of the proper flow-invariant subspace of dimension 1 fixed by $\kappa$ inside $\text{Fix}(\mathbb{Z}_2(\delta_3) \times \mathbb{Z}_2(\delta_1))$ and inside $\text{Fix}(\mathbb{Z}_2(\delta_1) \times \mathbb{Z}_2(\delta_2))$.

For solutions #5–#10, our calculations (with the parameter values investigated in this paper) show that the matrices are non-singular. Note that the right-hand sides of the matrix equations for cases 5–10 are nonzero, and so these solutions do not bifurcate directly from the Hopf/Hopf point, which is the generic case. The modulated standing waves (MSW) and the modulated rotating waves (MRW) emerge as secondary bifurcations from standing waves (SW) and rotating waves (RW) (see Chossat et al. $^{22}$ or Golubitsky et al. $^{23}$ for details).

Figures 6 and 7 show the $(q_a, q_r)$ parameter regions where the real solutions #0–#10 exist and are stable/unstable. (For the stability, we used Maple to calculate the eigenvalues $e_j$, $j = 1, 2, 3, 4$ of (3.19) corresponding to the steady states #0–#10.) For a direct comparison of the parameter regions where solutions exist...
Fig. 6. (Continued)
and the parameter regions where solutions are stable/unstable, we show solutions #0–#4 in Fig. 6 and solutions #5–#10 in Fig. 7.

As expected, the zero amplitude solution (#0) has its stability region determined by the neutral stability curves \( \Re(\sigma(k_3)) = 0 \) and \( \Re(\sigma(k_4)) = 0 \) (compare Fig. 6(a)′ with Fig. 2(b)). All other solutions are unstable in the parameter region around the H/H point where they exist, i.e. where \( \Lambda_j^2 \geq 0, j = 1, \ldots, 4 \) (see Figs. 6(b)–(e)′ and Figs. 7(a)–(f)′). In particular, this shows that the rotating waves (RW) and the standing waves (SW) bifurcating from the Hopf bifurcation curve bounding the stability region do so subcritically. This result is consistent with the numerical simulations, which show unstable spatio-temporal patterns around the 3:4 H/H point (see Fig. 5).
Regarding the RW #1 and #2 (Figs. 6(b) and 6(c)), we note that they cannot co-exist in the parameter region where the zero-amplitude solution is unstable (i.e. for large $q_a$ and small $q_r$, or for small $q_a$ and large $q_r$). Neither the SW #3 and #4 (Figs. 6(c) and 6(d)) can co-exist in the parameter region where this zero-amplitude solution is unstable. The MSW #5 can exist for large $q_a$ and small $q_r$ (see Fig. 7(a), where the two coloured regions overlap). Thus for large $q_a$ and small $q_r$, the MSW #5 bifurcates from the SW #3 (which exists in this region). Similarly, for small $q_a$ and large $q_r$, the MSW #6 bifurcates from the SW #4. The MRW #7 and #8 can exist only for small $q_a$ and small $q_r$ (see Figs. 7(c) and 7(d)). However, the numerical patterns shown in Figs. 5(d) and 5(d’) were obtained for large $q_a$ (i.e. $q_a \geq 5$).

These analytical and numerical results are not in contradiction, since the numerical patterns likely bifurcate secondary or tertiary from some rotating waves, and the weakly nonlinear analysis (which gives the cubic amplitude equations (3.19)) can predict only the primary bifurcations occurring near the bifurcation point.

Finally, solutions #9 and #10 also seem to exist only in a very restricted parameter space (where the coloured surfaces in Figs. 7(e) and 7(f) overlap). The fact that we obtained them numerically also in other regions (see Fig. 5) could mean that those numerical patterns are part of a solution branch that bifurcates in this region.

4. Summary and Discussion

In this paper we focused on a nonlocal hyperbolic model for biological aggregations, and investigated in detail the types of patterns that can arise near a codimension-2 Hopf/Hopf bifurcation point. Both the weakly nonlinear analysis and the numerical results showed that near the 3:4 Hopf/Hopf point, all patterns are unstable (i.e. bifurcate sub-critically). However, the weakly nonlinear analysis is valid only near the bifurcation point. Therefore, the results of the numerical simulations showing modulated rotating waves further away from the bifurcation point (i.e. for $q_a \geq 0.78$, $q_r = 3.56$), do not contradict the analytical results.

By investigating the parameter spaces where the various steady states of the amplitude equations can exist, we showed that the standing waves #3 exist in the same parameter space as the rotating waves #1, and the standing waves #4 exist in the same parameter space as the rotating waves #2. The modulated standing waves #5 and #6 can coexist in a small parameter space where $q_a \geq 0.63$ and $q_r \leq 3.6$, and where the state #0 is stable. Also, the modulated rotating waves #7–#10 can coexist in some narrow parameter regions, where they are unstable but the state #0 is stable (and the dynamics thus approaches a spatially homogeneous steady state). Thus, our analysis of model (2.1) identified the parameter spaces where multiple patterns/behaviours can exist simultaneously. Biologically, this means that for exactly the same parameter values (i.e. same speed, turning rates, magnitudes of social interactions), the groups can display different behaviours. Moreover, the transitions between different behaviours (e.g. from standing waves to rotating waves) do not involve changes in the parameter values that characterise the individual
movement or the magnitude of inter-individual interactions. Previous studies\(^1\) have shown that transitions between different aggregation patterns can be obtained by changing various parameters (e.g. the size of interaction zones, or group polarization\(^1\)). Our study supports the idea that in some cases, such transitions can also be intrinsic to the aggregations (and do not require changes in the parameter values).

As so many studies focus only on the bifurcation of various spatial and spatio-temporal patterns in biological systems, we emphasise here the importance of considering the model symmetries when deriving the amplitude equations. First, note that if we would have ignored the reflectional symmetries (i.e. \(\alpha_2 = \beta_2 = 0\) ), at \(O(\epsilon^3)\) we would have obtained a system of two coupled equations for the evolution of the amplitudes \(\alpha_1(T)\) and \(\beta_1(T)\) on the slow time scale. This system would have had only four steady states corresponding to homogeneous solutions (with zero amplitudes), two rotating wave solutions and standing wave solutions. Thus, we would have missed the modulated standing wave solutions and the modulated rotating wave (tori) solutions. Second, if we would have ignored the symmetries of the model when investigating the equilibria of the amplitude equations (3.19), we would have obtained a very large number of steady states. However, the symmetries of the model restricted the number of possible equilibria.

To conclude, in this study we investigated the symmetries and the bifurcation of spatio-temporal patterns displayed by a macroscopic model for self-organised biological aggregations. However, this model has been previously derived via a correlated random-walk approach from a microscopic model for the probabilities of left-moving and right-moving individuals.\(^\text{13}\) An individual-based model corresponding to this correlated random walk approach has been derived by Gerda de Vries (personal communication). A possible extension of the analysis performed in this paper could see an investigation of the symmetries of the individual-based model and a comparison with the symmetries of the macroscopic model.

Appendix A. Details of the Calculations

The coefficients that appear in the solution \(u_1^\pm\) (Eq. (3.11)) are:

\[
\begin{align*}
v_{11} &= \gamma k_3 - \omega_3, \quad v_{12} = \gamma k_3 + \omega_3, \quad v_{21} = \omega_3 + \gamma k_3, \quad v_{22} = \gamma k_3 - \omega_3, \\
w_{11} &= \gamma k_4 - \omega_4, \quad w_{12} = \omega_4 + \gamma k_4, \quad w_{21} = \gamma k_4 + \omega_4, \quad w_{22} = \gamma k_4 - \omega_4.
\end{align*}
\]

The coefficients of the solution \(\hat{u}\) of the adjoint linear system are:

\[
\begin{align*}
v_{11}^* &= -i\omega_3 + \gamma ik_3 + R_2\hat{K}^+(k_3) - R_2\hat{K}^-(k_3), \\
v_{12}^* &= i\omega_3 + \gamma ik_3 + R_2\hat{K}^+(k_3) - R_2\hat{K}^-(k_3), \\
v_{21}^* &= -i\omega_3 - \gamma ik_3 + R_2\hat{K}^-(k_3) - R_2\hat{K}^+(k_3), \\
v_{22}^* &= i\omega_3 - \gamma ik_3 + R_2\hat{K}^-(k_3) - R_2\hat{K}^+(k_3), \\
w_{11}^* &= -i\omega_4 + \gamma ik_4 + R_2\hat{K}^+(k_4) - R_2\hat{K}^-(k_4),
\end{align*}
\]
\[ w_{12}^* = i\omega_4 + \gamma ik_4 + R_2 \dot{K}^+(k_4) - R_2 \dot{K}^-(k_4), \]
\[ w_{21}^* = -i\omega_4 - \gamma ik_4 + R_2 \dot{K}^-(k_4) - R_2 \dot{K}^+(k_4), \]
\[ w_{22}^* = i\omega_4 - \gamma ik_4 + R_2 \dot{K}^-(k_4) - R_2 \dot{K}^+(k_4). \]

At \( O(\epsilon^2) \), the nonlinear operator is
\[
N_2 = \begin{pmatrix} R_1(K^- u_1^+ - K^+ u_1^-)(u_1^+ + u_1^-) \\ -R_1(K^- u_1^+ - K^+ u_1^-)(u_1^+ + u_1^-) \end{pmatrix}.
\]

Substituting the expressions for \( u_1^\pm \) into \( N_2 = (N_2^1, N_2^2)^T \), we obtain
\[
N_2^1 = Q^1\beta_1^2 + Q^2\beta_2^2 + Q^3\beta_3^2 + Q^4\beta_4^2 + Q^5\alpha_1\alpha_2 e^{2i\omega t} + Q^6\beta_1\beta_2 e^{2i\omega t} + Q^7\alpha_1\alpha_2 e^{2i\omega t} + Q^8\beta_1\beta_2 e^{2i\omega t} + Q^9\alpha_1 e^{2i\omega t} + 2i\omega_4 + Q^{10}\beta_1 e^{2i\omega t} + Q^{11}\beta_2 e^{2i\omega t} + Q^{12}\beta_1 e^{2i\omega t} + Q^{13}\beta_2 e^{2i\omega t} + Q^{14}\beta_1 e^{2i\omega t} + Q^{15}\beta_2 e^{2i\omega t} + Q^{16}\beta_1 e^{2i\omega t} + Q^{17}\beta_2 e^{2i\omega t} + Q^{18}\beta_1 e^{2i\omega t} + Q^{19}\beta_2 e^{2i\omega t},
\]
and \( N_2^2 = -N_2^1 \). Then, the coefficients \( G^j, j = 1, \ldots, 20 \) of \( u_2^\pm \) (Eq. (3.16)) are calculated by solving the following equations, with the operator \( L_{\omega_4, -k} \) given by (3.13):
\[
L_{0,0} G^n = Q^n, \quad n = 1, 2, 3, 4, \\
L_{2\omega_4, 0} G^n = Q^5, \quad L_{2\omega_4, 0} G^6 = Q^6, \quad L_{0, 2k_4} G^7 = Q^7, \quad L_{0, 2k_4} G^8 = Q^8, \\
L_{2\omega_4, -2k_4} G^9 = Q^9, \quad L_{2\omega_4, -2k_4} G^{10} = Q^{10}, \quad L_{2\omega_4, 2k_4} G^{11} = Q^{11}, \\
L_{2\omega_4, -2k_4} G^{12} = Q^{12}, \quad L_{2\omega_4, -2k_4} G^{13} = Q^{13}, \quad L_{2\omega_4, -2k_4} G^{14} = Q^{14}, \\
L_{2\omega_4, -2k_4} G^{15} = Q^{15}, \quad L_{2\omega_4, -2k_4} G^{16} = Q^{16}, \\
L_{2\omega_4, -2k_4} G^{17} = Q^{17}, \quad L_{2\omega_4, -2k_4} G^{18} = Q^{18}, \\
L_{2\omega_4, -2k_4} G^{19} = Q^{19}, \quad L_{2\omega_4, -2k_4} G^{20} = Q^{20}.
\]

At \( O(\epsilon^3) \), the nonlinear operators are
\[
N_3 = \begin{pmatrix} R_1(K^- u_1^+ - K^+ u_1^-)(u_1^+ + u_1^-) + R_1(K^- u_2^+ - K^+ u_2^-)(u_1^+ + u_1^-) \\ + S_1(K^+ u_1^+ - K^- u_1^-)^2(u_1^+ + u_1^-) \\ + 2T_2(K^- u_1^+ - K^+ u_1^-)(u_1^+ + u_1^-) \\ -R_1(K^- u_1^+ - K^+ u_1^-)(u_1^+ + u_1^-) - R_1(K^- u_2^+ - K^+ u_2^-)(u_1^+ + u_1^-) \\ - S_1(K^+ u_1^+ - K^- u_1^-)^2(u_1^+ + u_1^-) \\ - 2T_2(K^- u_1^+ - K^+ u_1^-)(u_1^+ + u_1^-) \end{pmatrix}.
\]
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and

\[
E_3 = \left( -\frac{\partial u_1^+}{\partial T} - R_2(K_{1}^{+} u_1^-) + R_2(K_{1}^{-} u_1^-), \right. \\
\left. -\frac{\partial u_1^-}{\partial T} + R_2(K_{1}^{+} u_1^-) - R_2(K_{1}^{-} u_1^-) \right).
\]

Constants \( R_1 \) and \( R_2 \) are defined by \((2.15)\). The new terms are: \( S_1 = \lambda_2 f''(0)/2, \)
\( T_1 = \lambda_2 f''(0)/6, \) \( T_2 = 3T_1 u^* \). Finally, kernels \( K_{\pm}^{\pm} = \nu_{\pm} K_{\pm}^{\pm} - \nu_{\pm} K_{\pm}^{\pm} \).

At \( O(\epsilon^3) \), the \( e^{i\omega_{\pm}x + ik_{\pm}x} \) terms that appear in the nonlinear operator
\( N_3 + E_3 = (N_3^1 + E_3^1, N_3^2 + E_3^2)^T \) are:

\[
N_3^1 + E_3^1 = e^{i\omega_{\pm}t + ik_{\pm}x} \left( \frac{d\alpha_1}{dt} F_1^1 - \alpha_1 F_1^2 + \alpha_1 |\alpha_1|^2 F_1^3 \right.
\]
\[\left. + \alpha_1 |\alpha_2|^2 F_2^1 + \alpha_1 |\beta_1|^2 F_1^3 + \alpha_1 |\beta_2|^2 F_2^3 \right) + e^{i\omega_{\pm}t - ik_{\pm}x} \left( -\frac{d\alpha_1}{dt} H_1^1 - \alpha_2 H_2^3 + \alpha_2 |\alpha_1|^2 H_1^3 \right.
\]
\[\left. + \alpha_2 |\alpha_2|^2 H_1^3 + \alpha_2 |\beta_1|^2 H_2^3 + \alpha_2 |\beta_2|^2 H_2^3 \right) + e^{i\omega_{\pm}t + ik_{\pm}x} \left( -\frac{d\beta_1}{dt} M_1^1 - \beta_1 M_1^2 + \beta_1 |\alpha_1|^2 M_1^3 \right.
\]
\[\left. + \beta_1 |\alpha_2|^2 M_2^1 + \beta_1 |\beta_1|^2 M_1^3 + \beta_1 |\beta_2|^2 M_2^3 \right) + e^{i\omega_{\pm}t - ik_{\pm}x} \left( -\frac{d\beta_2}{dt} N_1^1 - \beta_2 N_1^2 + \beta_2 |\alpha_1|^2 N_1^3 \right.
\]
\[\left. + \beta_2 |\alpha_2|^2 N_2^1 + \beta_2 |\beta_1|^2 N_1^3 + \beta_2 |\beta_2|^2 N_2^3 \right) + \text{c.c.,}
\]

\[
N_3^2 + E_3^2 = e^{i\omega_{\pm}t + ik_{\pm}x} \left( \frac{d\alpha_1}{dt} F_2^1 - \alpha_1 F_2^2 + \alpha_1 |\alpha_1|^2 F_2^3 \right.
\]
\[\left. + \alpha_1 |\alpha_2|^2 F_2^1 + \alpha_1 |\beta_1|^2 F_2^3 + \alpha_1 |\beta_2|^2 F_3^3 \right) + e^{i\omega_{\pm}t - ik_{\pm}x} \left( -\frac{d\alpha_1}{dt} H_2^1 - \alpha_2 H_2^2 + \alpha_2 |\alpha_1|^2 H_2^3 \right.
\]
\[\left. + \alpha_2 |\alpha_2|^2 H_2^3 + \alpha_2 |\beta_1|^2 H_2^3 + \alpha_2 |\beta_2|^2 H_2^3 \right) + e^{i\omega_{\pm}t + ik_{\pm}x} \left( -\frac{d\beta_1}{dt} M_2^1 - \beta_1 M_2^2 + \beta_1 |\alpha_1|^2 M_2^3 \right.
\]
\[\left. + \beta_1 |\alpha_2|^2 M_2^1 + \beta_1 |\beta_1|^2 M_2^3 + \beta_1 |\beta_2|^2 M_2^3 \right) \]
Finally, the coefficients of the amplitude equations (3.17) are given by

\[ X_1 = \frac{v_{11}^6 F_3^5 + v_{12} F_2^3}{v_{11}^6 F_1^4 + v_{12} F_2^3}, \quad X_2 = \frac{v_{11}^6 F_3^5 + v_{12} F_2^3}{v_{11}^6 F_1^4 + v_{12} F_2^3}, \quad X_3 = \frac{v_{11}^6 F_3^5 + v_{12} F_2^3}{v_{11}^6 F_1^4 + v_{12} F_2^3}, \]

\[ Y_1 = \frac{v_{21} H_2^2 + v_{22} H_2^2}{v_{21} H_1^4 + v_{22} H_2^2}, \quad Y_2 = \frac{v_{21} H_2^2 + v_{22} H_2^2}{v_{21} H_1^4 + v_{22} H_2^2}, \quad Y_3 = \frac{v_{21} H_2^2 + v_{22} H_2^2}{v_{21} H_1^4 + v_{22} H_2^2}, \]

\[ Z_1 = \frac{w_{11}^1 M_1^2 + w_{12} M_2}{w_{11}^1 M_1^2 + w_{12} M_2}, \quad Z_2 = \frac{w_{11}^1 M_1^2 + w_{12} M_2}{w_{11}^1 M_1^2 + w_{12} M_2}, \quad Z_3 = \frac{w_{11}^1 M_1^2 + w_{12} M_2}{w_{11}^1 M_1^2 + w_{12} M_2}, \]

\[ \Psi_1 = \frac{w_{21}^1 N_1^2 + w_{22}^1 N_2^2}{w_{21}^1 N_1^2 + w_{22}^1 N_2^2}, \quad \Psi_2 = \frac{w_{21}^1 N_1^2 + w_{22}^1 N_2^2}{w_{21}^1 N_1^2 + w_{22}^1 N_2^2}, \quad \Psi_3 = \frac{w_{21}^1 N_1^2 + w_{22}^1 N_2^2}{w_{21}^1 N_1^2 + w_{22}^1 N_2^2}. \]

Appendix B. Explicit Solutions for Cases #5–#10

- Case #5: \( \lambda_3 = \lambda_4 = 0 \) and

\[ (A_2)^2 = \frac{-\text{Re}(Y_1) + \text{Re}(Y_2) \left( \frac{\text{Re}(X_1)}{\text{Re}(X_2)} \right)}{-\text{Re}(Y_3) + \text{Re}(Y_2) \left( \frac{\text{Re}(X_1)}{\text{Re}(X_2)} \right)}, \quad (A_1)^2 = \frac{\text{Re}(X_1) - \lambda_2^2 \text{Re}(X_3)}{\text{Re}(X_2)} \]

- Case #6: \( \lambda_1 = \lambda_2 = 0 \) and

\[ (A_4)^2 = \frac{-\text{Re}(\Psi_1) + \text{Re}(\Psi_4) \left( \frac{\text{Re}(Z_1)}{\text{Re}(Z_2)} \right)}{-\text{Re}(\Psi_3) + \text{Re}(\Psi_4) \left( \frac{\text{Re}(Z_1)}{\text{Re}(Z_2)} \right)}, \quad (A_3)^2 = \frac{\text{Re}(Z_1) - \lambda_2^2 \text{Re}(Z_5)}{\text{Re}(Z_4)}. \]
Case #7: $\Lambda_1 = \Lambda_3 = 0$ and

$$ (A_4)^2 = \frac{-\text{Re}(\Psi_1) + \text{Re}(\Psi_3) \left( \frac{\text{Re}(Y_1)}{\text{Re}(Y_3)} \right)}{-\text{Re}(\Psi_5) + \text{Re}(\Psi_3) \left( \frac{\text{Re}(Y_3)}{\text{Re}(Y_3)} \right)}, \quad (A_2)^2 = \frac{\text{Re}(Y_1)}{\text{Re}(Y_3)} - (A_4)^2 \frac{\text{Re}(Y_3)}{\text{Re}(Y_3)}. $$

Case #8: $\Lambda_1 = \Lambda_4 = 0$ and

$$ (A_3)^2 = \frac{-\text{Re}(Z_1) + \text{Re}(Z_3) \left( \frac{\text{Re}(Y_1)}{\text{Re}(Y_3)} \right)}{-\text{Re}(Z_4) + \text{Re}(Z_3) \left( \frac{\text{Re}(Y_3)}{\text{Re}(Y_3)} \right)}, \quad (A_2)^2 = \frac{\text{Re}(Y_1)}{\text{Re}(Y_3)} - (A_3)^2 \frac{\text{Re}(Y_3)}{\text{Re}(Y_3)}. $$

Cases #9, #10: $\Lambda_1 = \Lambda_2, \Lambda_3 = \pm \Lambda_4$, with

$$ (A_3)^2 = \frac{\text{Re}(Z_1) - \text{Re}(X_1) \frac{\text{Re}(Z_3)+\text{Re}(Z_5)}{\text{Re}(X_3)+\text{Re}(X_3)}}{\text{Re}(Z_4) + \text{Re}(Z_5) - \frac{\text{Re}(Z_2)+\text{Re}(Z_3)\text{Re}(X_4)+\text{Re}(X_5)}{\text{Re}(X_4)+\text{Re}(X_5)}}, $$

$$ (A_1)^2 = \frac{\text{Re}(X_1)}{\text{Re}(X_2) + \text{Re}(X_3)} - (A_3)^2 \frac{\text{Re}(X_3) + \text{Re}(X_5)}{\text{Re}(X_2) + \text{Re}(X_3)}. $$

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