Symmetry-Breaking Bifurcations in Rings of Delay-Coupled Semiconductor Lasers

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Abstract. We consider symmetric rings of delay-coupled lasers modeled using the Lang–Kobayashi (LK) rate equations with unidirectional and bidirectional coupling. Because of phase symmetry the networks have symmetry groups $\mathbb{Z}_n \times \mathbb{S}^1$ (unidirectional) and $\mathbb{D}_n \times \mathbb{S}^1$ (bidirectional). Our first main result is a characterization of isotropy subgroups of those actions from which we determine sufficient conditions for the existence of basic classes of compound laser modes (CLMs) valid for all ring sizes. Case studies of the $n = 3$ and $n = 8$ coupled LK equations are presented, including branches of CLMs obtained via DDE-Biftool. Using the block diagonalization of the linearization coming from the isotypic decomposition we classify the symmetry-breaking type of steady-state and Hopf bifurcation points at fully symmetric CLMs and in the case $n = 3$ obtain explicit location of bifurcation points. We complement the study using DDE-Biftool to perform branch continuation at steady-state and Hopf bifurcation points in the cases $n = 3$ and $n = 8$. In particular, we obtain branches of periodic solutions, known as bridges, connecting branches of CLMs.

Key words. symmetry, bifurcations, lasers, Lang–Kobayashi equations, networks

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1. Introduction. We study a system of $n$ identical Lang–Kobayashi (LK) equations modeling semiconductor lasers, with nearest neighbor unidirectional coupling and nearest neighbor bidirectional coupling. We assume all optical coupling between the lasers and this introduces a time-delay in the reinjection of light from one laser to its neighbor(s). We assume the delay, the coupling strength, and the coupling phase to be identical for all coupling terms; therefore, the unidirectional coupling makes the system $\mathbb{Z}_n$ symmetric and the bidirectional coupling leads to a $\mathbb{D}_n$-symmetric network. The LK equations are rate equations describing the operation of a semiconductor laser and consist of one complex equation for the electric field and one equation for the inversion (corresponding to electron-holes). The LK equations are also symmetric with respect to rotations of the electric fields and so the system has an additional $\mathbb{S}^1$ symmetry, thus leading to networks with symmetry groups $\mathbb{Z}_n \times \mathbb{S}^1$ and $\mathbb{D}_n \times \mathbb{S}^1$.

Our main results are concerned with existence and bifurcation of so-called compound laser modes (CLM) solutions which are rotating wave periodic orbits where all the lasers operate at
the same optical frequency, but possibly with different amplitudes. The goal of this work is to generalize to \( n \) lasers certain aspects of the work of Erzgraber, Krauskopf, and Lenstra [11] who perform a bifurcation analysis of CLMs for two symmetrically coupled lasers with bidirectional coupling. Erzgraber, Krauskopf, and Lenstra [11] study analytically and using DDE-Biftool the appearance of CLMs, and obtain branches of synchronous, antiphase, and symmetry-broken CLMs with the coupling phase as continuation parameter. Forced symmetry-breaking of the system is done by continuing with respect to the detuning parameter between the lasers; stability and bifurcations from symmetric branches are obtained using DDE-Biftool. The goals of this paper are (1) to use group-theoretic methods to show how one can classify possible symmetric patterns of CLMs in ring networks of LK lasers with symmetry, (2) use theoretical tools from equivariant bifurcation theory for the computation of steady-state and Hopf bifurcations to complement the results obtained via automatic numerical continuation softwares, such as DDE-Biftool, and (3) begin case studies for three and eight lasers and obtain preliminary results using DDE-Biftool about bifurcation diagrams.

Coupled networks of lasers have been the subject of a very large literature since the advent of lasers. We only describe in more detail a selection of papers with a more direct relation to this one. For a nice and thorough introduction to single laser dynamics, but especially for its review of recent results on networks of lasers, see Soriano et al. [32]. Apart from the work in [11], other recent work on small networks of lasers include studying the stability of the zero-lag solution in three linearly coupled lasers; see [33] and [12]. More recently, Flunkert and Schöll [13] studied chaos synchronization using the master stability function in two-laser network motifs modeled via LK equations. D’Huys et al. [8] study symmetrically coupled phase oscillator systems with delay as a model for networks of lasers. In particular, they look at the case of unidirectionally coupled \( Z_n \) and bidirectionally coupled \( D_n \) symmetric networks where they compute the existence of frequency locked solutions and their stability. The Stuart–Landau equation approximates the dynamics of the LK equations and D’Huys et al. [7] investigate the effect of variations of the delay on symmetry-breaking bifurcations for \( Z_n \) and \( D_n \) symmetric rings of Stuart–Landau oscillators (with identical coupling phase). Small networks of Stuart–Landau oscillators are studied in [6] as a function of the coupling phase and compared to phase oscillator networks. They show that both phase and amplitude are important to discuss stability properties of solutions. In this paper, we go a step further by studying explicitly the dynamics of delay coupled rings of LK equations with \( Z_n \) and \( D_n \) symmetry and we also assume that the coupling phases are identical. Note that the dynamics of ring networks of lasers modeled via LK-type equations in nonsymmetric cases have been investigated by several authors [2, 13, 4, 17] and also for networks with other topologies [5, 22, 1]. The experimental realization of unidirectional and bidirectional ring networks of lasers is discussed in [4, 17, 3]. Moreover, in [1] the possibility of more complex topologies is introduced in the case of two-mode lasers.

The symmetry assumption on the networks is made for reasons of both modeling and mathematical analysis. By introducing the \( Z_n \) and \( D_n \) ring symmetry structure, we can study networks with arbitrarily large numbers of lasers, with the symmetry group structure acting as a unifying feature for each family of networks. Recently, the study of large networks of lasers/optical components has attracted attention. Nixon et al. [26] study networks with thousands of coupled (nearly identical) lasers over different lattice configurations and investi-
gate experimentally and theoretically the occurrence of geometric frustration in the network. Pecora et al. [27] consider synchronization and cluster synchronization of large networks of identical units bidirectionally coupled to other nodes. The resulting network has a symmetry group given by a subgroup of the full permutation group of the network which may not be obtained easily only by inspection. They infer the symmetry group from the cluster synchronization patterns obtained and use isotypic decomposition techniques of the linearization to study stability. They apply their results to an electro-optic network of light-emitting diodes.

Our approach to the study of CLMs is through a mixture of analytical and algebraic methods along with numerical computations and numerical continuation using DDE-Biftool. Because of the arbitrary size of the networks under study in this paper, we use group-theoretic techniques to classify CLMs according to isotropy subgroups via twisted subgroups of \( \mathbb{Z}_n \times \mathbb{S}^1 \) and \( \mathbb{D}_n \times \mathbb{S}^1 \); see [9]. This leads to several different synchronization patterns, many of which correspond to sublattice synchronized states [25, 3, 32]. CLMs are obtained by substituting an ansatz of solution into the network equations which yields a transcendental equation that one has to solve to find the frequency of the CLM. Because of the complexity of looking for CLMs in general, we focus our search to symmetric CLMs; that is, CLMs that are fixed by a nontrivial isotropy subgroup of the symmetry group of the equations. Since the characterization in terms of isotropy subgroups is valid for all \( n \), for a given isotropy subgroup we obtain a particular form of transcendental equation. We show that for certain maximal isotropy subgroups, CLMs exist for all \( n \). We illustrate the classification of isotropy subgroups and the derivation of transcendental equations with specific examples. We then proceed to study the symmetry-breaking bifurcations from maximally symmetric (\( \mathbb{Z}_n \) and \( \mathbb{D}_n \)) CLM solutions for both unidirectional and bidirectional networks. Because the rotating part of the CLM can be easily factored out, the linearization entries at any CLM are constants. The isotypic decomposition determined by the group actions is used to simplify the linearization at CLMs with \( \mathbb{Z}_n \) and \( \mathbb{D}_n \) symmetry. This allows us to use local equivariant bifurcation results from equilibria to obtain the bifurcating branches. In particular, the isotypic decomposition at both CLMs is computed and leads to a block diagonalization of the characteristic matrix and a decoupling of the characteristic equation; computations are similar to the ones in Chapter XVIII in [16]. Although DDE-Biftool can keep track of stability and bifurcations along branches, we show how to use the decoupled characteristic equation to confirm the bifurcation points obtained via DDE-Biftool and determine the symmetry type of the bifurcation points. Numerical simulations for the cases \( n = 3 \) and \( n = 8 \) are presented to illustrate the results.

This paper is organized as follows. Section 2 describes modeling assumptions behind the ring networks and the system of equations given by the Lang–Kobayashi models. Isotropy subgroups with the corresponding patterns are obtained for the \( \mathbb{Z}_n \times \mathbb{S}^1 \)-symmetric unidirectional network and \( \mathbb{D}_n \times \mathbb{S}^1 \)-symmetric bidirectional network. In section 3, we determine the transcendental equations corresponding to several cases of isotropy subgroups obtained in the previous section; in particular, for certain classes of maximal isotropy subgroups. We then present case studies with \( n = 3 \) and \( n = 8 \). Section 5 describes the linearization at CLMs with \( \mathbb{Z}_n \) and \( \mathbb{D}_n \) symmetry and we present the classification of symmetry-breaking steady-state and Hopf bifurcations. We then confirm the presence of the numerically computed bifurcation points and classify the symmetry groups. We conclude the section with more numerical simulations with DDE-Biftool.
2. Symmetric \( n \)-laser systems.

2.1. The systems of equations. We consider a mathematical model of a ring of \( n \) identical semiconductor lasers coupled to their neighbors via light injection in two configurations. In the unidirectional coupling configuration, laser \( j \) injects light in laser \( j + 1 \) (mod \( n \)) while in the bidirectional coupling configuration, the beam from laser \( j \) is injected into lasers \( j - 1 \) and \( j + 1 \) (mod \( n \)).

The assumption of identical lasers is common and has been used by several authors [22, 11, 14, 17, 2, 13] and obtaining nearly identical lasers for experiments is common [20]. We assume that the lasers are coupled identically with the coupling strength controlled by the amount of light transmitted between the lasers. Because of finite propagation time of light between the lasers, the coupling is a function of a fixed delay depending on the distance between the lasers. We can choose a unique delay if the distance between lasers is assumed identical. However, this requirement can be relaxed for unidirectional coupling as described further below. The light injection from a neighboring laser comes with a phase shift in the electrical field and we assume that the coupling phase is identical for all connections between lasers. From the assumption of identical frequencies of lasers, identical coupling phase is reasonable and a more detailed discussion of this assumption is found below. The assumption of identical coupling therefore depends on the coupling strength, the delay time, and the coupling phase.

Rate equations for two weakly-coupled semiconductor lasers are derived rigorously by Mulet, Masoller, and Mirasso [23] in the form of two Lang–Kobayashi equations coupled via an extra term depending on the delayed electric field of the other laser and also on a detuning term; the difference of each intrinsic frequency. We use this modeling approach using rate equations for our \( n \) laser ring networks by extending the model of two coupled lasers in [11] and adopt their notation. See, for instance, [21, 22, 4, 13] for a similar approach to modeling \( n \) coupled lasers.

We consider the case of \( n \) delay-coupled semiconductor identical lasers modeled using the Lang–Kobayashi equations. For a ring of lasers with unidirectional coupling we have the system of delay-differential equations (DDE)

\[
\begin{align*}
\dot{E}_j(t) &= (1 + i\alpha)N_j(t)E_j(t) + \kappa e^{-iC_p}E_{j-1}(t - \tau), \\
\dot{N}_j(t) &= \frac{1}{T}[P - N_j(t) - (1 + 2N_j(t))|E_j(t)|^2]
\end{align*}
\]

for \( j = 1, \ldots, n \) (mod \( n \)) where the parameters are the linewidth enhancement factor \( \alpha \), the coupling strength \( \kappa \), the coupling phase \( C_p \), the pump parameter \( P \), and \( T \) is the decay rate of the electrons. For a ring with bidirectional coupling we have the DDE

\[
\begin{align*}
\dot{E}_j(t) &= (1 + i\alpha)N_j(t)E_j(t) + \kappa e^{-iC_p}(E_{j-1}(t - \tau) + E_{j+1}(t - \tau)), \\
\dot{N}_j(t) &= \frac{1}{T}[P - N_j(t) - (1 + 2N_j(t))|E_j(t)|^2]
\end{align*}
\]

for \( j = 1, \ldots, n \) (mod \( n \)). Note that a unidirectional ring of \( n \) units with arbitrary delays \( \tau_1, \ldots, \tau_n \) between units can be transformed via a reparametrization of time to depend on a unique delay \( \tau = (\tau_1 + \cdots + \tau_n)/n \); see Perlikowski et al. [28]. Therefore, if the coupling phases are assumed to be the same or can be neglected, the applicability of our results about (1) can be mapped to the unidirectional ring case with delays \( \tau_1, \ldots, \tau_n \).
The coupling phase corresponds to the amount of rotation of the electric field as light travels from one laser to its neighbor and is thus a function of the delay time and the frequency of the light in the laser. Because we assume identical frequencies, say Ω, for all lasers, then $C_p = \Omega \tau$. Coupling phase can be controlled, for instance, by changing the distance between the lasers on the scale of the optical wavelength of the lasers in order to cover the range of values $[0, 2\pi]$. Note that changes in $C_p$ typically do not alter $\tau$ and those are often considered independent [19]. Although the assumption of identical coupling phase is possibly challenging to achieve experimentally for a large number of lasers, agreement between experiments and simulations for two symmetrically delay-coupled lasers (with identical coupling phase) is satisfying [20]. In any case, systems (1) and (2) can be used as a starting point to study the effect of different coupling phases and detuning between the lasers.

The two coupled laser system studied in [11] for lasers with frequencies $\Omega_1$ and $\Omega_2$ is given by

$$
\begin{align*}
\dot{E}_1(t) &= (1 + i\alpha)N_1(t)E_1(t) + \kappa e^{-iC_p}E_2(t - \tau) - i\Delta E_1(t), \\
\dot{E}_2(t) &= (1 + i\alpha)N_2(t)E_2(t) + \kappa e^{-iC_p}E_1(t - \tau) + i\Delta E_2(t), \\
\dot{N}_1(t) &= T^{-1}[P - N_1(t) - (1 + 2N_1(t))|E_1(t)|^2], \\
\dot{N}_2(t) &= T^{-1}[P - N_2(t) - (1 + 2N_2(t))|E_2(t)|^2],
\end{align*}
$$

where $\Delta = \frac{1}{2}(\Omega_2 - \Omega_1)$ and $C_p = \Omega \tau$ with $\Omega = \frac{1}{2}(\Omega_1 + \Omega_2)$. For $\Omega_1 = \Omega_2$ this system reduces to (1) for $n = 2$. We are not aware of a derivation as in [23] for more than two mutually coupled lasers. For unidirectional and bidirectional rings of lasers with nonidentical frequencies, several generalizations are possible which reduce to (1) and (2). For instance, let $\Omega_0$ be the average of the free-running frequencies $\Omega_i$ (for $i = 1, \ldots, n$) and $C_p = \Omega_0 \tau$. Then a possible model similar to [22] or [17] is given by

$$
\begin{align*}
\dot{E}_j(t) &= \tilde{\Omega}_i E_j + (1 + i\alpha)N_j(t)E_j(t) + \kappa e^{-iC_p}E_{j-1}(t - \tau), \\
\dot{N}_j(t) &= T^{-1}[P - N_j(t) - (1 + 2N_j(t))|E_j(t)|^2],
\end{align*}
$$

where $\tilde{\Omega}_i = \Omega_i - \Omega_0$ and similarly for the bidirectional ring.

### 2.2. Symmetry groups and isotropy subgroups.

We rewrite (1) and (2) as

$$
\dot{X}_j(t) = f(X_j(t)) + \sum_{k=1}^{n} C_{j,k}h(X_j(t), X_k(t - \tau))
$$

for $j = 1, \ldots, n$ and exhibit the symmetries of the system. Here, we take $X_j(t) = [E_j(t), N_j(t)]^T \in \mathbb{C} \times \mathbb{R}$ for $j = 1, \ldots, n$. The internal dynamics function $f$ and the coupling function $h$ are given as follows:

$$
\begin{align*}
f(X_j(t)) &= f\left(\begin{bmatrix} E_j(t) \\ N_j(t) \end{bmatrix}\right) = \frac{1}{T}(P - N_j(t) - (1 + 2N_j(t))|E_j(t)|^2), \\
h(X_j(t), X_k(t - \tau)) &= h\left(\begin{bmatrix} E_j(t) \\ N_j(t) \end{bmatrix}, \begin{bmatrix} E_k(t - \tau) \\ N_k(t - \tau) \end{bmatrix}\right) = \begin{bmatrix} \kappa e^{-iC_p}E_k(t - \tau) \\ 0 \end{bmatrix}.
\end{align*}
$$
Notice that only the coupling function \( h \) carries the delay. The connection matrix \( C := (C_{j,k}) \), whose entries are given by
\[
C_{j,k} = \begin{cases} 
1 & \text{if cells } j \text{ and } k \text{ are coupled,} \\
0 & \text{otherwise,}
\end{cases}
\]
depends on the coupling architecture of the laser system. In (1), laser \( j \) receives a feedback only from laser \( j - 1 \). Meanwhile, in (2) laser \( j \) receives feedback both from laser \( j - 1 \) and laser \( j + 1 \). Thus, the connection matrices for systems (1) and (2) are given, respectively, by the following \( n \times n \) circulant matrices:
\[
C_n = \begin{bmatrix} 
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 
\end{bmatrix}
\]
and \( C_n + C_n^T \).

Dionne, Golubitsky, and Stewart [9] study arrays of coupled identical cells whose symmetry group is a direct product of so-called global and internal (or local) symmetry groups; see [9] for details. The global symmetry group acts on the network while the internal symmetry group acts identically on all the cells. We now describe the general construction of the global and internal symmetry group for our ring networks.

Let \( X(t) = (X_1(t), \ldots, X_n(t)) \) and suppose the full system can be written abstractly as \( \dot{X}(t) = F(X(t), X(t - \tau)) \). The action of a permutation \( \rho \in S_n \) on the state variables is given by
\[
\rho.X(t) = (X_{\rho^{-1}(1)}(t), \ldots, X_{\rho^{-1}(n)}(t)).
\]
The global symmetry group \( G \) permutes the cells in the network, so \( G \subset S_n \). It then follows that for all \( \rho \in G \),
\[
F(\rho.X(t), \rho.X(t - \tau)) = \rho.F(X(t), X(t - \tau)).
\]
Let \( Z_n = \langle \gamma \rangle \), \( D_n = \langle \gamma, \mu \rangle \). We define the action of these groups on \( X(t) \) as permutation on the indices leading to
\[
\gamma.X_j(t) = X_{j+1}(t), \quad \mu.X_j(t) = X_{n+2-j}(t),
\]
where the indices are computed mod \( n \). A straightforward computation shows that \( G = Z_n \) and \( G = D_n \) are the global symmetry groups for systems (1) and (2). The group \( S^1 \) acts diagonally on \( X(t) \); that is, on each \( X_j = (E_j, N_j)^T \), with \( j = 1, \ldots, n \)
\[
\vartheta.(E_j, N_j)^T = (e^{i\vartheta} E_j, N_j)^T.
\]
One can check that for \( \vartheta \in S^1 \),
\[
f(\vartheta.X_j(t)) = \vartheta.f(X_j(t)).
\]
Therefore, the internal symmetry group is \( \mathcal{L} = S^1 \). Because
\[
h(\vartheta \cdot X_j(t), \vartheta \cdot X_k(t - \tau)) = \vartheta h(X_j(t), X_k(t - \tau)),
\]
then \( \vartheta \) also becomes a symmetry of the full system; see [9]. Therefore, (1) has symmetry group \( \mathbb{Z}_n \times S^1 \) while (2) has symmetry group \( D_n \times S^1 \).

We complete this discussion by noticing the following straightforward \( 2\pi \) translational symmetry imposed on \( R \)
\[
(E_1, \ldots, E_n, C_p) \mapsto (E_1, \ldots, E_n, C_p + 2\pi)
\]
and the less obvious combination of state and parameter symmetry given by
\[
(E_1, \ldots, E_n, C_p) \mapsto (E_1, e^{2\pi i/n} E_2, \ldots, e^{2(n-1)\pi i/n} E_n, C_p + 2\pi/n).
\]
In the case \( n = 2 \), this symmetry reduces to the \( \pi \)-translational symmetry described in [11]. We call it the \( 2\pi/n \)-translational symmetry.

The basic solutions of (1) and (2) are called \textit{compound laser modes} (CLMs) and are of the form
\[
E_j(t) = R_j e^{i\omega t + i\sigma_j}, \quad N_j(t) = N_j
\]
with \( \sigma_1 = 0, R_j > 0, N_j, \omega, \) and \( \sigma_j \) real-valued for \( j = 1, \ldots, n \). CLMs are rotating waves, that is, periodic orbits that are also orbits of the \( S^1 \)-symmetry. In the next sections, we classify the isotropy subgroups of \( \mathbb{Z}_n \times S^1 \) and \( D_n \times S^1 \). The isotropy subgroup of a CLM imposes conditions on \( R_j, N_j, \) and the phase shifts, and this leads to a classification of the CLMs according to their isotropy subgroups. We also exhibit in numerical simulations that the \( 2\pi/n \)-translational symmetry maps solutions to solutions with different (but possibly conjugate) isotropy subgroups.

\subsection*{2.3. Isotropy subgroups.} We are interested in solutions that are fixed by subgroups of \( \mathbb{Z}_n \times S^1 \) and \( D_n \times S^1 \), the so-called isotropy subgroups; see [16] for definition and details. We characterize the isotropy subgroups of the actions (5) and (6). We focus only on the action of \( \mathbb{Z}_n \times S^1 \) and \( D_n \times S^1 \) on \( E = (E_1, \ldots, E_n) \in \mathbb{C}^n \) because, with \( S^1 \) acting trivially on \( N = (N_1, \ldots, N_n) \in \mathbb{R}^n \), we can deduce a representative element of \( N \) from that of \( E \).

Note that it is straightforward to see that \((0, \ldots, 0) \in \mathbb{C}^n \) has isotropy subgroup \( \mathbb{Z}_n \times S^1 \) or \( D_n \times S^1 \). But the action of \( \mathbb{Z}_n \) or \( D_n \) fixes \( N \) if \( N_j = N_1 \) for \( j = 2, \ldots, n \). The point \((E, N_1, \ldots, N_n) = (0, P, \ldots, P) \) is the trivial equilibrium for systems (1) and (2) where all lasers are quiescent. This equilibrium is unstable and we do not consider it in what follows. We begin by introducing the notion of twisted subgroups of a group \( \Gamma \) from [16]. Let \( H \) be a subgroup of \( \Gamma \) and let \( \theta : H \to S^1 \) be a group homomorphism. We call
\[
H^\theta = \{(h, \theta(h)) \in \Gamma \times S^1 \mid h \in H\}
\]
a \textit{twisted subgroup} of \( \Gamma \times S^1 \).

We begin with the isotropy subgroups of the \( \mathbb{Z}_n \times S^1 \) action. Let \( k \) be an integer between 1 and \( n \), set \( d = \gcd(k, n) \), and consider the subgroup \( \mathbb{Z}_{n/d} \) of \( \mathbb{Z}_n \) generated by \( \gamma^k \) and denoted by
Suppose \( \gcd(\ldots) \) and we obtain then
\[ H \] subspaces where the indices are taken mod \( k \leq \) only the possible nonzero elements) and the action of \( \phi \) is written
\[ (9) \quad \gamma^k, (E_j, E_{j+d}, \ldots, E_{j+n-d}) = (E_{j+n-k}, E_{j+n-k+d}, \ldots, E_{j+n-d}, \ldots, E_{j+n-k-d}), \] where the indices are taken mod \( n \). The mapping \( \theta_\ell : H \to S^1 \) defined by
\[ \gamma^k \mapsto \frac{2\pi d\ell}{n} \] is a homomorphism and \( H^0_\ell \) is a twisted subgroup of \( \mathbb{Z}_n \times S^1 \). The isotropy subgroups are given in the next result.

**Proposition 2.1.** The isotropy subgroups of \( \mathbb{Z}_n \times S^1 \) are twisted subgroups \( H^0_\ell \) where \( 1 \leq k \leq n-1 \) and \( 0 \leq \ell \leq (n/d) - 1 \). For \( 1 \leq k_1 \neq k_2 \leq n-1 \), if \( \gcd(k_1, n) = \gcd(k_2, n) = 1 \), then \( H^0_{k_1} \simeq H^0_{k_2} \simeq \mathbb{Z}_n \).

**Proof.** Let \( \Sigma \) be an isotropy subgroup of \( \mathbb{Z}_n \times S^1 \) and let \( \Pi : \mathbb{Z}_n \times S^1 \to \mathbb{Z}_n \) be the canonical projection. If \( E \in \text{Fix}(\Sigma) \), then, \( (\sigma, \phi).E = E \) for all \( \sigma \in \Pi(\Sigma) \) along with some corresponding \( \phi \in S^1 \). Thus \( \sigma = \gamma^k \) for some \( k, 1 \leq k \leq n-1 \). For each \( \sigma_j \), using (9) along with the \( \phi \) action, \( (\sigma, \phi).E = E \) implies
\[ (10) \quad (e^{i\phi} E_j, e^{i\phi} E_{j+n-k}, \ldots, e^{i\phi} E_{j+n-d}, \ldots, e^{i\phi} E_{j+n-k-d}) = (E_j, E_{j+d}, \ldots, E_{j+n-d}), \] and we obtain \( E_j e^{i\phi/d} = E_j \) from the cyclic action of \( \gamma^k \) of order \( n/d \). This forces \( \phi = 2\pi d\ell/n \) for some \( \ell \) and we can restrict it so that \( 0 \leq \ell \leq (n/d) - 1 \). Therefore, \( \phi = \theta_\ell(\gamma^k) \) and \( \Sigma = H^0_\ell \).

Suppose \( \gcd(k_1, n) = \gcd(k_2, n) = 1 \), then the order of the generators \( \gamma^{k_1} \) and \( \gamma^{k_2} \) is \( n \).

From the proof of Proposition 2.1 we deduce that elements of \( \text{Fix}(H^0_\ell) \) are of the following form: \( \sigma_j \in F_j \) satisfies
\[ (11) \quad E_{j+sd} = E_j e^{2\pi iqs/n}, \quad 1 \leq s \leq (n/d - 1), \] where \( q \) is chosen so that
\[ (12) \quad pn + q(k - n) = d \] for the smallest \( p \in \mathbb{N} \). That solutions \((q, p) \in \mathbb{N} \cup \{0\} \) exist is guaranteed by Bezout’s identity since \( k - n \leq 0 \). This formula is obtained by noticing in (10) that
\[ E_{j+d} = e^{i\phi} E_{j+n-k+d} = e^{i\phi} E_{j+(\frac{n-k}{d}+1)d}, \] then choosing the \((\frac{n-k}{d}+1)\)th element in \((E_j, E_{j+d}, \ldots, E_{j+n-d})\) and using again (10) we have
\[ E_{j+(\frac{n-k}{d}+1)d} = e^{i\phi} E_{j+n-k+(\frac{n-k}{d}+1)d} = e^{i\phi} E_{j+(\frac{n-k}{d}+1)d}. \]
Iterating this process, we obtain
\[ E_{j+\left(\frac{(q-1)(n-k)}{d}+1\right)d} = e^{i\phi} E_{j+\left(\frac{q(n-k)}{d}+1\right)d} \]

The relationship between \( E_{j+d} \) and \( E_j \) is obtained from
\[ E_{j+d} = e^{q_i\phi} E_{j+\left(\frac{(a-k)}{d}+1\right)} \]

for \( q \) such that \((q(n-k)/d + 1)d\) is the least multiple of \( n \) (recall that the indices are taken mod \( n \)). We now illustrate with an example.

**Example 2.2.** We consider first the case \((\mathbb{Z}_d \times S^1)\). For \( k = 1 \) and \( \ell \) chosen from 0, \ldots, 7, elements of \( \text{Fix}(H_{1d}^q) \) have the form
\[ (E_1, E_1 e^{\pi i \ell/4}, E_1 e^{\pi i \ell/2}, E_1 e^{3\pi i \ell/4}, E_1 e^{5\pi i \ell/4}, E_1 e^{7\pi i \ell/4}) \]

For \( k = 3, 5, 7 \), we have elements of \( \text{Fix}(H_{3d}^q) \) related by (11). For example, \( k = 3 \) implies \( d = 1 \) and \( 8p - 5q = 1 \) holds for \( q = 3 \) and \( p = 2 \); therefore, elements of \( \text{Fix}(H_{3d}^q) \) have the form
\[ (E_1, E_1 e^{3\pi i \ell/4}, E_1 e^{3\pi i \ell/2}, E_1 e^{\pi i \ell/4}, E_1 e^{\pi i \ell/2}, E_1 e^{3\pi i \ell/4}, E_1 e^{5\pi i \ell/4}, E_1 e^{7\pi i \ell/4}) \]

For \( k = 2 \), there are two subspaces \( F_j \) for \( j = 1, 2 \) with
\[ (E_j, E_{j+2}, E_{j+4}, E_{j+6}) = (E_j, E_j e^{\pi i \ell/2}, E_j e^{\pi i \ell}, E_j e^{3\pi i \ell/2}) \]

and \( \ell = 0, 1, 2, \text{ or } 3 \). For \( k = 4 \) there are four subspaces \( F_j \) with \( O_j = (E_j, E_{j+d}) \) for \( j = 1, 2, 3, 4 \) and \( \ell = 0, 1 \) only. Then, \((E_j, E_{j+1}) = (E_j, (-1)^\ell E_j)\) for \( \ell = 0, 1 \). Finally, for \( k = 6 \) we have \( d = 2 \) and so \( 8p - 2q = 2 \) is satisfied for \( q = 3 \) and \( p = 1 \) and one can easily write the representative element.

We now turn to the \( D_n \times S^1 \) case which has a richer structure. Let \( \epsilon = 0 \) or 1 and define
\[ \nu_\epsilon : \mathbb{Z}_2(\mu) \to S^1, \quad \nu_\epsilon(\mu) = e^{\epsilon \pi} \]

Note that if the only nonzero component of \( E \) is \( E_1 \), then \( \mu \) fixes \( E \) and for \( n \) even, this is true also if \( E_{1+n/2} \) is nonzero. Consider the subgroup \( D_{n/d} \cong \langle \gamma^k, \mu \rangle \) of \( D_n \) and let \( (D_{n/d})^{\theta_e, \nu_\epsilon} \subset D_n \times S^1 \) be an associated twisted subgroup.

**Proposition 2.3.** The isotropy subgroups of \( D_n \times S^1 \) are the twisted subgroups \( (D_{n/d})^{\theta_e, \nu_\epsilon} \).

If \( \ell = 0 \) or if \( \ell \neq 0 \) and \( k, q, \) and \( \ell \) are such that \( \ell n = 2dq \) for some \( p \in \mathbb{N} \), then \( (D_{n/d})^{\theta_e, \nu_\epsilon} \) is a dihedral group, otherwise it is a cyclic group.

**Proof.** Let \( \Sigma \) be an isotropy subgroup of \( D_n \times S^1 \) and let again \((\sigma, \phi) \in \Sigma \). Without loss of generality, we can restrict to checking the cases \( \sigma = \gamma^k \) and \( \sigma = \mu \). We consider the \( \sigma \)-invariant subspaces \( F_j \) with representatives \( O_j \) for \( j = 1, \ldots, d \). For \( \ell = 0 \), \( O_j = (E_j, \ldots, E_j) \).

If \((\mu, \epsilon) O_j = O_j \), then \( E_j = e^{\epsilon \pi} E_j \). If \( \epsilon = 1 \), then \( E_j = 0 \). Therefore, \( \Sigma = (D_{n/d})^{\theta_0, \nu_0} \).

For \( \ell \neq 0 \), we know from Proposition 2.1 that
\[ O_j = (E_j, E_{j+d}, \ldots, E_{j+(n/d-1)d}) \]
If \( j = 1 \), condition \((\mu, \epsilon)\). \( \mathcal{O}_1 = \mathcal{O}_1 \) implies for \( \epsilon = 1 \) that \( E_1 = 0 \). If \( \epsilon = 0 \), then the action of \( \mu \) on \( \mathcal{O}_1 \) implies

\[
E_1 \exp(2\pi idq\ell/n) = E_1 \exp(2\pi idq(n/d - 1)\ell/n)
\]

with \( q \) chosen as in (12). Therefore, \( \exp(4\pi idq\ell/n) = 1 \) and so \( pn = 2dq\ell \) for some \( p \in \mathbb{N} \). We determine the form of representative elements \( \mathcal{O}_j \) using formula (11). If \( p \) is even, then \( \mathcal{O}_1 = (E_1, \ldots, E_1) \), and if \( p \) is odd, then for \( j = 1 \)

\[
\mathcal{O}_j = (E_j, -E_j, \ldots, E_j, -E_j).
\]

If \( j \neq 1 \) and \( \mathcal{O}_j \) is fixed by \((\mu, \epsilon)\), then

\[
E_j = e^{i\pi}E_j \exp(2\pi idq(n/d - 1)\ell/n).
\]

This equality can be satisfied for \( E_j \neq 0 \) if \( \epsilon = 0 \) and \( pn = 2dq\ell \) with \( p \) even or \( \epsilon = 1 \) and \( pn = 2dq\ell \) with \( p \) odd. In the first case, \( \mathcal{O}_j = (E_j, \ldots, E_j) \) and in the second case \( \mathcal{O}_j \) has the form (15).

Suppose now that \( \mu \) interchanges elements of \( \mathcal{O}_j \) with \( \mathcal{O}_i \) for \( i \neq j \) where

\[
E_i = e^{i\pi}E_j \exp(2\pi idq(n/d - 1)\ell/n) \quad \text{and} \quad E_j = e^{i\pi}E_i \exp(2\pi idq(n/d - 1)\ell/n).
\]

Again, we must have \( pn = 2dq\ell \) for some \( p \in \mathbb{N} \). If \( p \) is even, then \( \mathcal{O}_j = (E_j, \ldots, E_j) \) and for \( p \) odd, \( \mathcal{O}_j \) is given by (15). The representative \( \mathcal{O}_i \) is obtained in terms of \( \mathcal{O}_j \) using \( E_i = e^{i\pi}E_j \exp(-p\pi i) \). Finally, the isotropy subgroups of \( \mathbb{Z}_n \times S^1 \) that do not satisfy the \( k \), \( q \), and \( \ell \) conditions above are also isotropy subgroups of \( D_n \times S^1 \).

3. Finding CLMs of symmetric \( n \)-laser systems. Each isotropy subgroup fixes a certain form of CLM; the isotropy subgroup specifies the values of the phase shifts \( \sigma_j \), and provides relations among \( R_j \) and \( N_j \). Substituting the ansatz (8) into systems (1) and (2) leads to a transcendental equation in the frequency \( \omega \) of the rotating wave. Solving this transcendental equation gives us the CLMs. We now show a general procedure to obtain those CLMs. We begin with the unidirectional coupling case for which we can get a general formula for CLMs, not necessarily with constant phase difference.

3.1. Unidirectional case. Substituting (8) into (1), we split real and imaginary parts, and simplify to obtain

\[
0 = N_j R_j + \kappa R_{j-1} \cos(C_p + \omega \tau + \sigma_j - \sigma_{j-1}),
\]

\[
\omega R_j = \alpha N_j R_j - \kappa R_{j-1} \sin(C_p + \omega \tau + \sigma_j - \sigma_{j-1}),
\]

\[
0 = P - N_j - (1 + 2N_j) R_j^2.
\]

Solving for \( N_j \) in the first equation, we have

\[
N_j = -\kappa \frac{R_{j-1}}{R_j} \cos(C_p + \omega \tau + \sigma_j - \sigma_{j-1})
\]
and substituting in the second equation we obtain

\[ R_j = \frac{-\kappa}{\omega}R_{j-1}[\alpha \cos(C_p + \omega \tau + \sigma_j - \sigma_{j-1}) + \sin(C_p + \omega \tau + \sigma_j - \sigma_{j-1})]. \tag{19} \]

The term in square brackets simplifies to \( \sqrt{1 + \alpha^2} \sin(C_p + \omega \tau + \sigma_j - \sigma_{j-1} + \arctan(\alpha)) \) and we set \( \theta_j := C_p + \omega \tau + \sigma_j - \sigma_{j-1} + \arctan(\alpha) \) and \( \kappa = \kappa \sqrt{1 + \alpha^2} \). Finally, starting from \( R_n \) we use (19) to write \( R_N \) in terms of \( R_{n-1} \), and iterate this process until we can write \( R_1 \) in terms of \( R_n \). From this, we derive

\[ \omega^n = (-1)^n \kappa^n \sin \theta_n \cdots \sin \theta_1 \]

and we have the following result.

**Proposition 3.1.** For a fixed set of dephasing constants \( \sigma_j, j = 1, \ldots, n \), if \( \omega \) satisfies (20), then \( E_j(t) = R_j e^{i\omega t + i\sigma_j} \) and \( N_j(t) = N_j \) is a CLM such that

\[ N_j = \frac{\omega}{\alpha + \tan(C_p + \omega \tau + \sigma_j - \sigma_{j-1})} \quad \text{and} \quad R_j = \sqrt{\frac{P - N_j}{1 + 2N_j}}. \tag{21} \]

**Proof.** The expression for \( N_j \) in (21) is obtained by first isolating \(-\kappa R_{j-1}/R_j\) from (18).

Then, we divide the second equation of (17) by \( R_j \) and substitute the \(-\kappa R_{j-1}/R_j\) value from (18) to obtain \( \omega = (\alpha + \tan(C_p + \omega \tau + \sigma_j - \sigma_{j-1}))N_j \). The \( R_j \) is obtained by solving the third equation of (17).

Expression (20) has a particularly simple form in some cases of CLMs. The following is reminiscent of the Equivariant Branching lemma in the fact that existence of the solution can be determined easily for maximal isotropy subgroups of \( \mathbb{Z}_n \times S^1 \).

**Proposition 3.2.** CLMs with maximal isotropy subgroup \( H^0_1 \) exist for all \( \ell \).

**Proof.** If \( k = 1 \), then the subspace \( F_1 \) has representative \( O = (E_1, \ldots, E_n) \) with \( E_j = E_1 e^{2\pi i(j-1)\ell/n} \) and so \( \sigma_j - \sigma_{j-1} = 2\pi \ell/n \) for all \( j \). This means \( \theta_j \) is constant for all \( j \). Equation (20) becomes \( \omega^n - (-1)^n \kappa^n \sin^n \theta_1 = 0 \) for which the left-hand side has a factor \( \omega \pm \kappa \sin(C_p + \omega \tau + 2\pi \ell/n + \arctan(\alpha)) \), depending on the parity of \( n \). This factor can always be solved for \( \omega \neq 0 \) as the intersection of a line of slope 1 with a sine function with a nonzero dephasing. This gives a CLM with isotropy subgroup \( H^0_1 \) from Proposition 3.1.

### 3.2. Bidirectional case.

Substituting (8) into (2), we obtain after splitting real and imaginary parts (and simplifying)

\[ 0 = N_j R_j + \kappa(R_{j-1} \cos(C_p + \omega \tau + \sigma_j - \sigma_{j-1}) + R_{j+1} \cos(C_p + \omega \tau + \sigma_j - \sigma_{j+1})), \]

\[ \omega R_j = \alpha N_j R_j - \kappa(R_{j-1} \sin(C_p + \omega \tau + \sigma_j - \sigma_{j-1}) + R_{j+1} \sin(C_p + \omega \tau + \sigma_j - \sigma_{j+1})), \]

\[ 0 = P - N_j - (1 + 2N_j)R_j^2. \tag{22} \]

Again after solving for \( N_j \) in the first equation and substituting in the second, one obtains

\[ \omega R_j = -\kappa(R_{j-1}(\alpha \cos(C_p + \omega \tau + \sigma_j - \sigma_{j-1}) + \sin(C_p + \omega \tau + \sigma_j - \sigma_{j-1})) \]

\[ + R_{j+1}(\alpha \cos(C_p + \omega \tau + \sigma_j - \sigma_{j+1}) + \sin(C_p + \omega \tau + \sigma_j - \sigma_{j+1}))). \tag{23} \]

The bidirectional case cannot be solved in complete generality as the unidirectional one. However, some special cases simplify enough to be worked out for all \( n \). This is the case if
\( \Sigma \subset D_n \times S^1 \) is an isotropy subgroup as given in Proposition 2.3 such that the phase differences \( \sigma_j - \sigma_{j-1} \) are constant for all \( j \) or \( \sigma_j - \sigma_{j-1} \) is \( \pm \) a constant depending on the parity of \( j \). We look at cases where this happens for \( d = 1 \) and \( d = 2 \).

If \( d = 1 \), from (11) we have \( E_j = E_{j-1} e^{2\pi i q \ell/n} \). Thus, letting \( E_j = R_j e^{i\sigma_j + \sigma} \), we obtain \( R_j e^{i\sigma_j} = R_{j-1} e^{i\sigma_{j-1} + 2\pi iq \ell/n} \). Setting \( R_j = R_1 \) for all \( j = 2, \ldots, n \), this equation yields \( \sigma_j - \sigma_{j-1} = 2\pi i q \ell/n \) for some \( \ell \) in \( 0, \ldots, n-1 \). For \( n \) even and \( d = 2 \), \( \gamma^k \) decomposes \( C^n \) into two subspaces \( F_1 \) and \( F_2 \) of dimension \( C^n/2 \). If \( \ell = 0 \) and \( \epsilon = 0 \), then elements of \( \text{Fix}(D_{n/2}(\gamma^k, \mu)) \) have the form \((E_1, E_2, E_1, E_2, \ldots, E_1, E_2)\). Choosing \( E_1 = R_1 \) and \( E_2 = R_1 e^{i\alpha_2} \) with \( \sigma_2 \) arbitrary, then we have phase differences \( \sigma_j - \sigma_{j-1} = (-1)^j \sigma_2 \). We do not go further in determining conditions leading to these types of phase differences. Instead, we return to the question of existence of CLMs using this knowledge with the following result.

**Proposition 3.3.** Let \( \Sigma \subset D_n \times S^1 \) be an isotropy subgroup as given in Proposition 2.3. If \( \Sigma \) is such that we can choose \( R_j = R_1 \) for all \( j = 2, \ldots, n \) and the phase difference \( \sigma_j - \sigma_{j-1} \) is constant and different from \( \pm \pi/2 \) or \( \sigma_j - \sigma_{j-1} \) is \( \pm \) a constant depending on the parity of \( j \), then there exists a CLM with isotropy subgroup \( \Sigma \). In particular, this holds if \( \Sigma \) is a maximal isotropy subgroup such as \( D_n(\gamma, \mu) \) or \( Z_n(\gamma^k, 2\pi q/n) \) with \( d = 1 \) and \( q \) chosen as in (12) or the submaximal isotropy subgroup \( D_{n/2}(\gamma^k, \mu) \) with \( d = 2 \).

**Proof.** If \( \sigma_j - \sigma_{j-1} = \beta \) is constant for all \( j = 1, \ldots, n \), the system (23) can be rewritten as

\[
(24) \quad \omega R_j = -\kappa (R_{j-1} \sin(C_p + \omega \tau + \beta + \arctan(\alpha)) + R_{j+1} \sin(C_p + \omega \tau - \beta + \arctan(\alpha))), \quad j = 1, \ldots, n,
\]

where we defined earlier that \( \kappa = \sqrt{1 + \sigma^2} \). Consider now the case \( \sigma_j - \sigma_{j-1} = (-1)^j \beta \), then the system of equation is written

\[
(25) \quad \omega R_j = -\kappa (R_{j-1} + R_{j+1}) \sin(C_p + \omega \tau + \beta + \arctan(\alpha)), \quad j = 1, \ldots, n.
\]

Letting \( R = (R_1, \ldots, R_n)^T \), the linear systems (24) and (25) can be written as \( M_1 R = 0 \) and \( M_2 R = 0 \), where

\[
M_1 = \omega I_n + \kappa \sin(C_p + \omega \tau + \beta + \arctan(\alpha))C_n + \kappa \sin(C_p + \omega \tau - \beta + \arctan(\alpha))C_n^T
\]

and for the second system

\[
M_2 = \omega I_n + \kappa \sin(C_p + \omega \tau + \beta + \arctan(\alpha))(C_n + C_n^T).
\]

The matrix \( M_1 \) has eigenvalue

\[
\Lambda_1 := \omega + \kappa \sin(C_p + \omega \tau + \beta + \arctan(\alpha)) + \kappa \sin(C_p + \omega \tau - \beta + \arctan(\alpha))
\]

with eigenvector \((1, \ldots, 1)\) while the matrix \( M_2 \) has eigenvalue

\[
\Lambda_2 := \omega + 2\kappa \sin(C_p + \omega \tau + \beta + \arctan(\alpha))
\]

also with eigenvector \((1, \ldots, 1)\). Therefore, \( M_1 R = 0 \) has a nonzero solution if \( \Lambda_1 = 0 \). The equation \( \Lambda_1 = 0 \) has a solution for \( \omega > 0 \) if \( \beta \neq \pm \pi/2 \) and as long as \( C_p + \arctan(\alpha) > 0 \);
this can be obtained for \( C_p \) large enough. The system \( M_2 R = 0 \) has a nonzero solution if
\( \Lambda_2 = 0 \). In this case, there is a solution with \( \omega > 0 \) if \( C_p + \beta + \arctan(\alpha) > 0 \) and this is again realizable for \( C_p \) large enough. The discussion in the paragraph before the statement guarantees the existence of CLMs for the subgroups listed.

**Remark 3.4.** If one sets \( R_j = R_1 \) for \( j = 1, \ldots, n \), then nonzero solutions to system \( M_1 R = 0 \) must be in eigenspaces of zero eigenvalues of \( M_1 \). Note that the \( M_1 \) matrix is \( \mathbb{Z}_n \)-equivariant, but not \( \mathbf{D}_n \)-equivariant. Therefore, from the decomposition of \( \mathbb{C}^n \) into irreducible representations, real eigenvalues generically appear only for the trivial representation of \( \mathbb{Z}_n \) and the alternating representation of \( \mathbb{Z}_n \) if \( n \) is even. If \( \beta = \pm \pi/2 \), then the calculation above shows that we do not find a CLM with \( \omega > 0 \) in the trivial representation eigenspace and it is a straightforward computation to show that the same holds in the alternating representation eigenspace.

The proof of Proposition 3.3 leads to the finding that for zero dephasing between neighboring units, the frequency of the CLM is invariant with respect to the size of the network.

**Corollary 3.5.** A CLM with isotropy subgroup \( \mathbf{D}_n(\gamma, \mu) \) with frequency \( \omega \) satisfying
\[
\omega + 2\kappa \sin(C_p + \omega t + \arctan(\alpha)) = 0
\]
exists for all \( n \).

Note that there are more (maximal) isotropy subgroups for which CLMs can be shown to exist, but it is not our goal in this paper to provide an exhaustive list for all \( \mathbf{D}_n \) symmetric rings.

4. **Case studies: \( n = 3 \) and \( n = 8 \).** We illustrate the results of the previous subsections by looking specifically at cases \( n = 3 \) and \( n = 8 \) for both unidirectional and bidirectional coupling. In each case, we discuss the classification of isotropy subgroups, obtain the transcendental equations for CLMs, and exhibit numerical simulations of CLMs using DDE-Biftool.

4.1. **Three lasers.** We begin by using Propositions 2.1 and 2.3 to classify nontrivial isotropy subgroups and their orbit representatives. An isotropy subgroup of \( \mathbb{Z}_3 \times \mathbb{S}^1 \) is given by \( k = 1 \) and \( \ell = 1 \); that is, \( H_1^{(k)} = \mathbb{Z}_3(\gamma, 2\pi/3) \) fixes \( (E_1, E_1, E_1) \). For \( k = 1, 2 \), then \( \gcd(k, 3) = 1 \) and setting \( \ell = 0 \), then the other isotropy subgroup is \( H_2^{(k)} = \mathbb{Z}_3(\gamma^k) \) which fixes \( E = (E_1, E_1, E_1) \).

We now turn to \( \mathbf{D}_3 \times \mathbb{S}^1 \). Recall the definition of \( \nu_\epsilon \) for \( \epsilon = 0 \) or 1; see (13). We let \( \ell = 0 \) and \( k = 1, 2 \), then from the cyclic case \( \gamma^k \) fixes \( (E_1, E_1, E_1) \). For \( \epsilon = 0 \), \( \mathbf{D}_3(\gamma^k, \mu) \) fixes \( (E_1, E_1, E_1) \) with \( E_1 \neq 0 \). Note that \( \mathbf{D}_3(\gamma, \mu) \) and \( \mathbf{D}_3(\gamma^2, \mu) \) are conjugate. If \( \epsilon = 1 \), then \( E_1 = 0 \). If \( \ell = 1 \), then \( k = 3 \) and we have only two cases: \( \mathbb{Z}_2(\mu) \) (i.e., \( \epsilon = 0 \)) which fixes \( (E_1, E_2, E_2) \) and \( \mathbb{Z}_2(\mu, \pi) \) (i.e., \( \epsilon = 1 \)) which fixes \( (0, E_2, -E_2) \). The lattice of isotropy subgroups is given in Figure 1 and Table 1 has a summary of the representatives of fixed point subspaces for the isotropy subgroups for both \( E \) and \( N \).

Consider the bidirectional ring with \( \mathbf{D}_3 \times \mathbb{S}^1 \) symmetry. From Proposition 3.3, there exist CLMs with isotropy subgroups \( \mathbf{D}_3(\gamma, \mu) \) and \( \mathbb{Z}_3(\gamma, 2\pi/3) \). For \( \mathbb{Z}_2(\mu, \pi) \), orbit representatives have the form \( (0, E_2, -E_2, N_1, N_2, N_2) \), which means \( \dot{E}_1 = 0 \) and this case reduces to the two laser cases that are studied in [11] and a CLM is shown to exist. There remains to check the case \( \mathbb{Z}_2(\mu) \) which has orbit representative \( (E_1, E_2, E_2, N_1, N_2, N_2) \), with \( E_1 = R_1 e^{i\omega t} \) and
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\[ E_2 = R_2 e^{i \omega t + i \sigma_2} \]

In this case, one can show that if \( \omega > 0 \) is a solution of the transcendental equation

\[
\omega^2 + \omega \tilde{\kappa} \sin(\theta) - 2 \tilde{\kappa}^2 \sin(\theta - \sigma^2) \sin(\theta + \sigma^2) = 0,
\]

where \( \theta = C_p + \omega \tau + \arctan(\alpha) \), then the system of equations (21) for \( j = 1, 2, 3 \) has a solution \( (R_1, R_2, R_3) \) with \( R_1 \neq 0 \) and \( R_2 = R_3 \neq 0 \).

We exhibit branches of CLMs obtained via DDE-Biftool [10, 31]. We set the parameters as in [11]:

\[
\alpha = 2.5, \quad \kappa = 0.1, \quad C_p = 10, \quad \tau = 20, \quad T = 392, \quad P = 0.23.
\]

We obtain a branch of CLMs by varying \( C_p \) using a “starting point” given by finding a frequency \( \omega \) from the transcendental equations obtained from the system of equations (21).

For instance, \( \omega + 2 \tilde{\kappa} \sin(C_p + \omega \tau + \arctan(\alpha)) = 0 \) for \( D_3(\gamma, \mu) \)-symmetric CLMs or (26) for \( Z_2(\mu) \)-symmetric CLMs. In Figure 2 (left) we show two ellipses of CLMs, the largest one has isotropy subgroup \( D_3(\gamma, \mu) \) and the inner one has isotopy subgroup \( H_{\theta}^1 \). In Figure 2 (right), we plot the branch of CLMs in \( C_p N \) plane for \( H_{\theta}^1 \) for \( \ell = 0 \), \( \ell = 1 \), and \( \ell = 2 \). The branches \( \ell = 1 \) and \( \ell = 2 \) are \( \pm 2\pi/3 \) translated from the \( H_{\theta}^1 \) branch. Observe also the \( 2\pi \)-translational symmetry in \( C_p \) and the \( C_p - 2\pi/3 \) translational symmetry permuting the branches \( \ell = 0, 1, 2 \).

For the case \( Z_2(\mu) \), \( \sigma_2 = \sigma_3 \) could be any nonzero real number, and we take the value \( \sigma_2 = \pi/6 \) and the solutions indeed satisfy \( R_1 \neq 0 \) and \( R_2 = R_3 \). See Figure 3.

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**Table 1**

Isotropy subgroups of \( D_3 \times S^1 \) and orbit representative of the fixed-point subspace.

<table>
<thead>
<tr>
<th>Isotropy subgroup</th>
<th>Orbit representative</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Z_2(\mu) )</td>
<td>( (E_1, E_2, E_3, N_1, N_2, N_3) )</td>
</tr>
<tr>
<td>( Z_3(\gamma, \frac{2\pi}{3}) )</td>
<td>( (E_1, E_2 e^{2\pi i/3}, E_1 e^{4\pi i/3}, N_1, N_1) )</td>
</tr>
<tr>
<td>( Z_2(\mu, \pi) )</td>
<td>( (0, E_2, -E_2, N_1, N_2, N_2) )</td>
</tr>
<tr>
<td>( D_3(\gamma, \mu) )</td>
<td>( (E_1, E_1, E_1, N_1, N_1, N_1) )</td>
</tr>
</tbody>
</table>
4.2. Eight lasers. For this example, we describe how to use the propositions to find isotropy subgroups, but do not provide the isotropy lattice which is quite large and cumbersome. In the case \( n = 8 \) for \( k = 1, 3, 5, 7 \), \( \gcd(k, 8) = 1 \) and the isotropy subgroups of \( \mathbb{Z}_8 \times S^1 \) are given by \( \mathbb{Z}_8(\gamma^k, \pi q\ell/4) \) with \( q \) chosen as in (12) for \( 0 \leq \ell \leq 7 \) and they are all conjugate for a fixed \( \ell \). For \( \ell = 1, 3, 5, 6, 7 \), \( k \) and \( \ell \) do not satisfy the \( k, \ell \) conditions of Proposition 2.3 which implies \( \mathbb{Z}_8(\gamma^k, \pi q\ell/4) \) are isotropy subgroups of \( D_8 \times S^1 \) and fix elements \( E = (E_1, e^{i\pi q\ell/4}E_1, \ldots, e^{i\pi q\ell/4}E_1) \). For \( \ell = 0 \), we have \( E_j = E_1 \) for \( j = 2, \ldots, 8 \) and so \( E \) is fixed by \( D_8(\gamma, \mu) \). Consider the remaining cases of \( k, q, \) and \( \ell \) such that \( 8p = 2dq\ell \) for some \( p \in \mathbb{N} \). For \( k = 1 \), then \( q = 1 \), and \( 8p = 2\ell \) is satisfied with \( \ell = 4 \) and \( p = 1 \). Thus, we have
$D_8((\gamma, \pi), \mu)$ is an isotropy subgroup and fixes

$$(E_1, -E_1, E_1, -E_1, E_1, -E_1, E_1, -E_1).$$

For $k = 2$, then $q = 1$, and $8p = 4\ell$ is satisfied with $\ell = 0$ and $p = 0$. Thus, we have $D_4(\gamma^2, \mu)$ fixing

$$(E_1, E_2, E_1, E_2, E_1, E_2, E_1, E_2).$$

Another solution of $8p = 4\ell$ holds for $p = 1$ and $\ell = 2$, the isotropy subgroups are $D_4((\gamma^2, \pi), \mu)$, and $D_4((\gamma^2, \pi), (\mu, \pi))$, fixing, respectively,

$$(E_1, 0, -E_1, 0, E_1, 0, -E_1, 0) \quad \text{and} \quad (0, E_2, 0, -E_2, 0, E_2, 0, -E_2).$$

For $k = 4$, then $q = 1$ and $8p = 8\ell$ has a solution for $p = 1$ and $\ell = 1$. Thus, we have $D_2((\gamma^4, \pi), \mu)$ and $D_2((\gamma^4, \pi), (\mu, \pi))$ which fix

$$(E_1, E_2, E_3, -E_2, -E_1, -E_2, -E_3, E_2) \quad \text{and} \quad (0, E_2, E_3, 0, -E_2, -E_3, -E_2).$$

We obtain as isotropy subgroups, $Z_8(\gamma, \pi/2)$ fixing $(E_1, iE_1, -E_1, -iE_1, E_1, iE_1, -E_1, -iE_1)$ and $Z_4(\gamma^2, \pi/2)$ fixing the element $(E_1, E_2, iE_1, iE_2, -E_1, -E_2, -iE_1, -iE_2)$. Finally, for $k = 8, q = 1$, and $\ell = 1$, then $Z_2(\mu)$ and $Z_2(\mu, \pi)$ fix, respectively,

$$(E_1, E_2, E_3, E_4, E_5, E_4, E_3, E_2) \quad \text{and} \quad (E_1, E_2, E_3, E_4, E_5, -E_4, -E_3, -E_2).$$

In the unidirectional case, Proposition 3.2 guarantees that CLMs with maximal isotropy subgroup $H_1^\theta$ exist for all $\ell$. We illustrate the case of the submaximal subgroup $H_2^0$ which has orbit representative $(E_1, E_2, -E_1, -E_2, E_1, E_2, -E_1, -E_2)$. Let $E_1 = R_1 e^{i\sigma_1}$ and $E_2 = R_2 e^{i\sigma_2}$. Because $\sigma_1 = 0$, the angles $\theta_j = C_p + \omega \tau + \arctan(\alpha) + \sigma_j - \sigma_{j-1}$ are

$$\theta_1 = \theta_5 = C_p + \omega \tau + \arctan(\alpha) - \sigma_2 - \pi,$$

$$\theta_2 = \theta_4 = \theta_6 = \theta_8 = C_p + \omega \tau + \arctan(\alpha) + \sigma_2,$$

$$\theta_3 = \theta_7 = C_p + \omega \tau + \arctan(\alpha) + \pi - \sigma_2.$$

Thus, $\sin \theta_1 = \sin \theta_5 = \sin \theta_3 = \sin \theta_7$. The transcendental equation is

$$\omega^8 - \tilde{\kappa}^8 \sin^4 \theta_1 \sin^4 \theta_2 = 0$$

which we can rearrange to

$$(\omega^2 - \tilde{\kappa}^2 \sin \theta_1 \sin \theta_2)(\omega^2 + \tilde{\kappa}^2 \sin \theta_1 \sin \theta_2)(\omega^2 + (\tilde{\kappa}^4 \sin \theta_1 \sin \theta_2)^2) = 0.$$

The last factor is nonzero. The first two factors can be solved by noticing that

$$\sin \theta_1 \sin \theta_2 = \frac{1}{2} \left( \cos(2(\omega \tau + C_p + \arctan(\alpha))) - \cos(2\sigma_2) \right)$$

and this leads to

$$\cos(2(\omega \tau + C_p + \arctan(\alpha))) = \cos 2\sigma_2 \pm \frac{2\omega^2}{\tilde{\kappa}^2}.$$
The right-hand side of the equations is a parabola in $\omega$ and one can verify that it always has an intersection with the curve on the left-hand side for some $\omega \neq 0$ for an open set of values of $\bar{\kappa}$, $C_p$, $\tau$, and $\sigma_2$. Therefore, one should expect to encounter CLMs with submaximal symmetry groups.

In the $D_8$-symmetric bidirectional case, apart from the subgroups listed in Proposition 3.3, there is also a CLM with symmetry $D_8((\gamma, \pi), \mu)$ because the hypotheses of the proposition are satisfied. We look for some of the CLMs not previously obtained from Proposition 3.3. For $D_4((\gamma^2, \pi), (\mu, \pi))$ the orbit representative

$$(0, E_2, 0, -E_2, 0, E_2, 0, -E_2)$$

leads to a case where $\dot{E}_i = 0$ for $i = 1, 3, 5, 7$ and this reduces to a ring of four lasers for which one can show the existence of a CLM because of the constant phase differences. A similar situation of reduction to four lasers occurs for $D_4((\gamma^2, \pi), \mu)$. Finally, $Z_8(\gamma, \pi/2)$ with orbit representative $(E_1, iE_1, -E_1, -iE_1, E_1, iE_1, -E_1, -iE_1)$ has constant phase shifts of $\pi/2$ and so by Proposition 3.3 and Remark 3.4, generically there are no CLMs with this isotropy subgroup. The other cases lead to higher degree transcendental equations and we omit them.

Numerical simulations are presented in section 5.6, including bifurcations.

5. Linearization and bifurcations. The rotating wave character of a CLM can easily be transformed to an equilibrium solution using polar coordinates and substituting the expression for the CLM. This amounts to “freezing” the rotation of the CLM by rotating at the same frequency as the CLM. Therefore, we can use local bifurcation results for equilibria on CLMs. In this section, we classify the symmetry groups of the bifurcating branches of CLMs from steady-state and Hopf bifurcation points using the equivariant branching lemma and the equivariant Hopf theorem. Those bifurcating branches of CLMs are then followed with DDE-Biftool. DDE-Biftool is capable of locating bifurcation points and then following them to branches of CLMs. However, it is not possible to identify the type of symmetry-breaking occurring from the bifurcation point. This is important since for $D_n$-symmetric systems, multiple branches of solutions bifurcate; see [16]. We address this issue by using the symmetry of the CLM to reduce the characteristic matrix corresponding to the linear variational equation of the $n$-laser system around the CLM to a block diagonal form. Computing the characteristic equation reduces to computing the determinant of each diagonal block and we show how to alternatively obtain the location of bifurcation points from each block.

5.1. Block diagonalization. We begin with the block diagonalization from the linearization of the system at fully symmetric CLMs for both unidirectional and bidirectional coupling: CLMs with isotropy subgroups $H_i^{\psi} \simeq \mathbb{Z}_n$ and $D_n((\gamma, \mu))$. In both cases, $E_i = E_j$ and $N_i = N_j$ for all $i, j = 1, \ldots, n$.

5.2. Symmetric $n$-laser system with unidirectional coupling. For a symmetric system of $n$ lasers with unidirectional coupling, the rate equations for the $j$th laser in polar form, for
\[ \hat{R}_j(t) = N_j(t)R_j(t) + \kappa \lambda_{j-1}(t - \tau) \cos(-C_p + \varphi_{j-1}(t - \tau) - \varphi_j(t)), \]
\[ \hat{\varphi}_j(t) = \alpha N_j(t) + \kappa \frac{R_{j-1}(t - \tau)}{R_j(t)} \sin(-C_p + \varphi_{j-1}(t - \tau) - \varphi_j(t)), \]
\[ \hat{N}_j(t) = \frac{1}{T} [P - N_j(t) - (1 + 2N_j(t))|R_j(t)|^2]. \]

If we let \( X_j(t) = [R_j(t), \varphi_j(t), N_j(t)]^T \) and \( Y_{j-1}(t) = X_{j-1}(t - \tau) \), then (27) can be written in the form \( \dot{X}_j(t) = f(X_j(t), Y_{j-1}(t)) \), for \( j = 1, \ldots, n \). Letting \( X(t) = [X_1(t), X_2(t), \ldots, X_n(t)]^T \) and \( Y(t) = [Y_1(t), Y_2(t), \ldots, Y_n(t)]^T \), the full system can be written as \( \dot{X}(t) = F(X(t), Y(t)) \).

To obtain the linear variational equation around a CLM for the full system, we first compute \( \bar{A} := d_{X_j(t)}f(\text{CLM}) \) and \( \bar{B} := d_{Y_k(t)}f(\text{CLM}) \), where CLM is given by

\[ (R_j(t), \varphi_j(t), N_j(t)) = (R, \omega t, N) \]
for \( j = 1, \ldots, n \). Note that for \( R \neq 0 \) and \( \varphi = \omega t \), \( N = -\kappa \cos(C_p + \omega \tau) \) solves the first equation of (27). We obtain

\[ \bar{A} = \begin{bmatrix} -c & -R s & R \\ \frac{\bar{s}}{\bar{\tau}} & -c & \alpha \\ -c_2 & 0 & -c_1 \end{bmatrix} \quad \text{and} \quad \bar{B} = \begin{bmatrix} c & R s & 0 \\ -\frac{\bar{s}}{\bar{\tau}} & c & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]

where

\[ s = \kappa \sin(C_p + \omega \tau), \quad c = \kappa \cos(C_p + \omega \tau), \]
\[ c_1 = \frac{1}{T} (1 + 2R^2), \quad c_2 = \frac{\bar{s}}{\bar{\tau}} (1 + 2N)R. \]

Let \( M_1 = I_n \otimes \bar{A} \) and \( M_2 = C_n \otimes \bar{B} \). Then, \( dF(\text{CLM}) = [M_1|M_2] \), and the linear variational equation around the CLM (28) is given by \( \ddot{X}(t) = M_1X(t) + M_2X(t - \tau) \). The characteristic equation is \( \det(\Delta(\lambda)) = 0 \), where \( \Delta(\lambda) = \lambda I - M_1 - e^{-\lambda \tau} M_2 \). Notice that if we let \( A = \lambda I - \bar{A} \) and \( B = -e^{-\lambda \tau} \bar{B} \), then

\[ L = \Delta(\lambda) = I_n \otimes A + C_n \otimes B. \]

Our goal now is to reduce \( L \) into a block diagonal form. Following the procedure in [16, pp. 390–391], and using \( \zeta = \exp(2\pi i/n) \), We consider the subspaces

\[ V_k = \{ [1, \zeta^k, \zeta^{2k}, \ldots, \zeta^{(n-1)k}]^T v \mid v \in \mathbb{C}^3 \} \]

for \( k = 0, 1, \ldots, n - 1 \) which decomposes into three isomorphic copies of \( \mathbb{Z}_n \) irreducible representations. Notice that \( L|_{V_0} = I_n \otimes (A + B) \) while if \( n \) is even, then \( L|_{V_{n/2}} = I_n \otimes (A - B) \). Moreover, for \( V_k \) with \( k > 1 \) we have \( L|_{V_k} = I_n \otimes (A + \zeta^{(n-1)k}B) \) and identifying \( V_k \simeq \mathbb{R}^{6n} \), then

\[ L|_{V_k} = I_n \otimes \begin{bmatrix} A + \text{Re}(\zeta^{(n-1)k}B) & -\text{Im}(\zeta^{(n-1)k}B) \\ \text{Im}(\zeta^{(n-1)k}B) & A + \text{Re}(\zeta^{(n-1)k}B) \end{bmatrix}. \]
Thus, for \( n \) even, \( L = \text{diag}(A + B, A - B, \Gamma_1, \ldots, \Gamma_{\frac{n}{2} - 1}) \) where

\[
\Gamma_k = \begin{bmatrix}
A + \text{Re}(\zeta^{(n-1)k})B & -\text{Im}(\zeta^{(n-1)k})B \\
\text{Im}(\zeta^{(n-1)k})B & A + \text{Re}(\zeta^{(n-1)k})B
\end{bmatrix}
\]

for \( k = 1, \ldots, n - 1 \). When \( n \) is odd, we only have \( L = \text{diag}(A + B, \Gamma_1, \ldots, \Gamma_{(n-1)/2}) \), where \( \Gamma_k \) is as in (30) for \( k = 1, \ldots, (n - 1)/2 \). Since \( L = \Delta(\lambda) \) can be written in a block diagonal form, the problem of solving the characteristic equation \( \det \Delta(\lambda) = 0 \) reduces to solving the equations \( \det(A + B) = 0 \), \( \det(A - B) = 0 \), and \( \det(\Gamma_k) = 0 \).

5.3. Symmetric \( n \)-laser system with bidirectional coupling. For the system of \( n \) lasers with bidirectional coupling, the rate equations for the \( j \)th laser in polar form, for \( j = 1, \ldots, n \), is given by

\[
\dot{R}_j(t) = N_j(t)R_j(t) + \kappa \left[ \sum_{k=1, k \neq j}^{j+1} \frac{R_k(t-\tau) \cos(-C_p + \varphi_k(t-\tau) - \varphi_j(t))}{R_j(t)} \right],
\]

\[
\dot{\varphi}_j(t) = \alpha N_j(t) + \kappa \left[ \sum_{k=1, k \neq j}^{j+1} \frac{R_k(t-\tau)}{R_j(t)} \sin(-C_p + \varphi_k(t-\tau) - \varphi_j(t)) \right],
\]

\[
\dot{N}_j(t) = \frac{1}{T} |P - N_j(t) - (1 + 2N_j(t))|R_j(t)|^2|.
\]

Using the same notation as in the previous subsection, (31) can be written in a form \( \dot{X}_j(t) = f(X_j(t), Y_1(t), Y_2(t), \ldots, Y_{j-1}(t), Y_{j+1}(t), \ldots, Y_n(t)) \), where \( j = 1, \ldots, n \). To obtain the linear variational equation for the full system around the CLM in (28), we first compute for \( \bar{A} := dX_{j(t)}f(\text{CLM}) \) and \( \bar{B} := d_{Y_{j(t)}}f(\text{CLM}) \), where CLM is as in (28). We get

\[
\bar{A} = \begin{bmatrix}
-2c & -2Rs & R \\
-2c & -c_2 & \alpha \\
-c_2 & 0 & -c_1
\end{bmatrix}
\text{ and } \bar{B} = \begin{bmatrix}
c & Rs & 0 \\
-\frac{R}{c} & c & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Then, \( dF(\text{CLM}) = [M_1 \mid (C_n + C_n^T) \otimes \bar{B}] \), and the linear variational equation around the CLM (28) is given by \( \dot{X}(t) = M_1 X(t) + ((C_n + C_n^T) \otimes \bar{B}) X(t - \tau) \). The characteristic equation for this linear variational equation is \( \det \Delta(\lambda) = 0 \), where \( \Delta(\lambda) = \lambda I_n - M_1 - e^{-\lambda \tau}((C_n + C_n^T) \otimes \bar{B}) \). Notice that if we let \( A = \lambda I_3 - \bar{A} \) and \( B = -e^{-\lambda \tau} \bar{B} \), then

\[
L = \Delta(\lambda) = I_n \otimes A + (C_n + C_n^T) \otimes B.
\]

This block matrix form has already been examined in [16, p. 395]. Their results are as follows. Using the same \( \zeta \) and \( \varphi_k \) as in the unidirectional case, for \( k = 0, \ldots, n - 1 \), we have \( L|_{V_k} = I_n \otimes (A + (\zeta^k + \zeta^{(n-1)k})B) = I_n \otimes (A + \ell_k B) \), where \( \ell_k = 2 \cos(2\pi k/n) \). The above
information can be used to reduce $L$ into a block diagonal form. For example, when $n = 3$, $L = \text{diag}(A + 2B, A - B, A - B)$. Since $L = \Delta(\lambda)$ can be written in a block diagonal form

$$\Delta(\lambda) = \text{diag}(A + 2B, A + \ell_1 B, \ldots, A + \ell_{n-1} B),$$

the problem of solving the characteristic equation $\det \Delta(\lambda) = 0$, reduces to solving the equations $\det(A + \ell_k B) = 0$ for $k = 0, \ldots, n-1$. Note, however, that $\ell_j = \ell_{n-j}$ for $j \neq 0$ and the block diagonalization differs for $n$ odd and $n$ even. For $n$ odd we have

$$\Delta(\lambda) = \text{diag}(A + 2B, A + \ell_1 B, A + \ell_1 B, \ldots, A + \ell_{[n/2]} B, A + \ell_{[n/2]} B),$$

where $\lfloor \cdot \rfloor$ is the floor function and for $n$ even

$$\Delta(\lambda) = \text{diag}(A + 2B, A + \ell_1 B, A + \ell_1 B, \ldots, A + \ell_{[(n-1)/2]} B, A + \ell_{[(n-1)/2]} B, A - 2B).$$

### 5.4. Classifying bifurcations.

From the linearizations obtained in the previous section, we classify bifurcating branches of steady-state and periodic solutions bifurcating from CLMs using the equivariant branching lemma [16] and the equivariant Hopf theorem for DDEs [18]. Note that the equivariant branching lemma is stated for ODEs, but steady-state bifurcations in DDEs is identical to the ODE case since the delay has no effect.

Consider (27) and (31) as one-parameter families of DDEs depending on $C_p$. Suppose that there exists a parameter value such that the characteristic equation has a zero eigenvalue; that is, $\ker \Delta(0) \neq \{0\}$. The equivariant branching lemma states that for every isotropy subgroup $H \subset G$ with a one-dimensional fixed-point subspace in $\ker \Delta(0)$, if the eigenvalue crosses the imaginary axis with nonzero speed, there exists a branch of steady-state solutions with isotropy subgroup $H$ bifurcating from $X_0$.

For unidirectionally coupled networks, the subspaces $V_0$ and $V_n$ for $n$ even are the only ones which are direct sums of absolutely irreducible representations of the $Z_n$ action. Therefore, the zero eigenvalue is generically from the $A + B$ or the $A - B$ block. In the case of the $A + B$ block, either $X_0$ undergoes a saddle-node bifurcation or a transcritical bifurcation and the symmetry group is unchanged. If $n$ is even and the zero eigenvalue is from the $A - B$ block, then $-I$ acts on $\ker \Delta(0)$ and the bifurcating branch has $Z_2(-I)$ symmetry, hence $X_0$ is a pitchfork bifurcation point and the bifurcating branch has isotropy subgroup $Z_{n/2}(\gamma^2)$. This is summarized in the following statement.

**Proposition 5.1.** Let $X_0$ be a CLM with isotropy subgroup $H_1^{(0)} \simeq \mathbb{Z}_n$ of a one-parameter family of unidirectionally coupled lasers given by (1). Then, generically, $X_0$ can undergo a steady-state bifurcation from the diagonal blocks $A + B$ if $n$ is odd and $A + B$ and $A - B$ if $n$ is even. Moreover, the bifurcating branches are as follows.

1. If the zero eigenvalue is from $A + B$, then $X_0$ undergoes either a saddle-node bifurcation or a transcritical bifurcation, in which case the bifurcating branch also has isotropy subgroup $H_1^{(0)}$.

2. If the zero eigenvalue is from $A - B$, then $X_0$ undergoes a pitchfork bifurcation and the only bifurcating branch has isotropy subgroup isomorphic to $Z_{n/2}(\gamma^2)$.

For bidirectionally coupled networks, we need to look at all the cases $V_k$ for $k = 0, \ldots, n-1$ as all of them are direct sums of absolutely irreducible representations of $D_n$. If $k = 0$, the
action is trivial and only saddle-node bifurcations can occur. For $k = n/2$, the action of $\gamma$ on $V_{n/2}$ is by $-1$ and $\mu$ acts trivially. For the other cases, this is a direct application of the equivariant branching lemma. The effective action of $D_n$ on $V_k$ is given by $D_m$ with $m = n/gcd(k, n)$. The only fixed point subspaces of dimension one for absolutely irreducible representations of $D_m$ with $m$ even are given by isotropy types $Z_2(\mu)$ and $Z_2(\gamma \mu)$ and for $m$ odd by isotropy types $Z_2(\mu)$; see [16] for details. We summarize in the following statement.

**Proposition 5.2.** Let $X_0$ be a CLM with isotropy subgroup $D_n(\gamma, \mu)$ of a one-parameter family of bidirectionally coupled lasers given by (2). Then, generically, $X_0$ can undergo a steady-state bifurcation from any diagonal block in $\Delta(\lambda)$. Moreover, the bifurcating branches are as follows.

1. If the zero eigenvalue is from $A + 2B$, then $X_0$ undergoes a saddle-node bifurcation and no new branch bifurcates.

2. If the zero eigenvalue is from $A - 2B$ ($n$ is even), then $X_0$ undergoes a pitchfork bifurcation and the only bifurcating branch of CLM has isotropy subgroup $D_{n/2}(\gamma^2, \mu)$.

3. If the zero eigenvalue is from $A - \ell_k B$ with $k > 0$ and $n/gcd(k)$ is odd, then a branch of CLM bifurcates for each isotropy subgroup conjugate to $Z_2(\mu)$.

4. If the zero eigenvalues from $A - \ell_k B$ with $k \neq 0, n/2$ and $n/gcd(k, n)$ is even, then a branch of CLM bifurcates for each isotropy subgroup conjugate to $Z_2(\mu)$ and $Z_2(\gamma \mu)$.

For CLMs $H_1^{\mu_0}$ of the unidirectional network and $D_n(\gamma, \mu)$ of the bidirectional network, there are no restrictions on obtaining purely imaginary eigenvalues from any block of the linearization. One can then apply the equivariant Hopf theorem for DDEs, and consider isotropy subgroups of $H_1^{\mu_0} \times S^1$ and $D_n(\gamma, \mu) \times S^1$ with two-dimensional fixed point subspaces, where now $S^1$ is the phase shift symmetry on periodic solutions. Branches of bifurcating solutions from steady-state and Hopf bifurcation are obtained below in the case $n = 3$ and in the unidirectional coupling case with $n = 8$.

**5.5. Symmetry-breaking bifurcation points.** Numerical continuation with DDE-Biftool identifies bifurcation points and their types. In this section, we use the block diagonal structure of the linearization to confirm in an independent way the location of the bifurcation points using a computer algebra package. Moreover, this has the advantage of determining the type of symmetry-breaking occurring according to Propositions 5.1 and 5.2. We illustrate this method on the branch of CLMs with $D_3(\gamma, \mu)$ symmetry.

**Steady-state bifurcations.** The determinant $|A - B|$ is given by

$$\lambda^3 + (4c + c_1)\lambda^2 + (Rc_2 + 4cc_1 + 4\kappa^2)\lambda + (2Rc_2(-\alpha s + c) + 4\kappa^2c_1)$$

$$+(2\kappa^2 + (4\alpha c_1 + 4\kappa^2)\lambda + Rc_2(-\alpha s + c) + 4\kappa^2c_1)e^{-\lambda \tau} + (\lambda + c_1)\kappa^2 e^{-2\lambda \tau}.$$

Observe that for $\lambda = 0$, $|A - B| = 3Rc_2(-\alpha s + c) + 9\kappa^2c_1$ which further simplifies to

$$|A - B|_{\lambda = 0} = \left(\frac{3\kappa}{T}\right) \left(2(P - N)[-\alpha \sin(C_p + \omega \tau) + \cos(C_p + \omega \tau)] + 3\kappa \frac{1 + 2P}{1 + 2N}\right)$$

after substituting the identities for $c_1$ and $c_2$ from (29), and using the identity $(1 + 2N)R^2 = P - N$ obtained from the third equation of (31). Notice that $|A - B| = 0$ depends on $\omega$ and $C_p$. From Proposition 5.1 we know that pitchfork bifurcations arise from this block and we
can obtain those by looking at the intersections, in $\omega C_p$-plane, of the curves $|A - B|_{\lambda=0} = 0$, and

$$\omega + 2\kappa \sqrt{1 + \alpha^2} \sin(C_p + \omega \tau + \arctan(\alpha)) = 0,$$

which is the transcendental equation corresponding to the isotropy subgroup $D_3(\gamma, \mu)$. These two curves are shown in Figure 4 (left), where the blue curve is the equation $|A - B|_{\lambda=0} = 0$, and the red curve is (32). There are only two intersections modulo $2\pi$ in $C_p$, $(C_p, \omega) \approx (3.2977, 0.2734)$ and $(4.6617, -0.5126)$. Thus, we only have two pitchfork bifurcation points coming from the $(A - B)$ block.

**Figure 4.** Left: Pitchfork bifurcation points from the $A - B$ block are found by looking at the intersections of the blue and red curves. Right: Saddle-node bifurcation points from the $A + 2B$ block, the approximate values are in the text.

We have $|A + 2B|_{\lambda=0} = 0$ because of the zero eigenvalue along the group orbit. To obtain the bifurcation points, we compute (after using identities as above)

$$\frac{d}{d\lambda}|A + 2B|_{\lambda=0} = \frac{2}{T}(P - N)(1 + 2\tau(-\alpha s + c)) = 0.$$

The intersection of this curve with (32) is shown in Figure 4 (right). Close inspection reveals intersection points at $(C_p, \omega) \approx (4.8765, -0.5385)$, and $(C_p, \omega) \approx (5.3093, 0.5385)$ modulo $2\pi$.

**Hopf bifurcations.** For Hopf bifurcation, again we split the equation $|A + 2B|_{\lambda=i\beta} = 0$ into real and imaginary parts. We then solve for $C_p$ in (32), just like in the previous case, and then substitute this value to the equations $\text{Re}|A + 2B|_{\lambda=i\beta} = 0$ and $\text{Im}|A + 2B|_{\lambda=i\beta} = 0$. Finally, we plot these two equations on the $\omega \beta$ plane giving us the red and blue curves in Figure 5. The intersections of these curves give us the Hopf bifurcation points on the branch of CLMs. In total, we get two Hopf bifurcation points from the $(A + 2B)$ block.

To look for Hopf bifurcation points, we set $\lambda = i\beta$. Equation $|A - B|_{\lambda=i\beta} = 0$ is a complex equation that we split into $\text{Re}|A - B|_{\lambda=i\beta} = 0$ and $\text{Im}|A - B|_{\lambda=i\beta} = 0$. We look for values
Figure 5. Standard Hopf bifurcation points are found by looking at the intersections of $\text{Re}|A + 2B|_{\lambda=i\beta} = 0$ (red) and $\text{Im}|A + 2B|_{\lambda=i\beta} = 0$ (blue). The curves at the left use $C_p$ values given in (33) while the curves at the right use values of $C_p$ in (34).

of $\beta$ and $\omega$ that give Hopf bifurcation points on the branch of CLMs. We first solve for $C_p$ in (32) and then substitute this value to $\text{Re}|A - B|_{\lambda=i\beta} = 0$ and $\text{Im}|A - B|_{\lambda=i\beta} = 0$. These two equations now depend on $\beta$ and $\omega$. Plotting them in the $\omega$-$\beta$ plane gives two curves whose intersections are the Hopf bifurcation points on the branch of CLMs. Figure 6 (left) shows these two curves for $C_p$ from (32), given by

\begin{equation}
C_p = \arcsin \left( -\frac{\omega}{2\kappa \sqrt{1 + \alpha^2}} - \omega \tau - \arctan(\alpha) \right)
\end{equation}

while Figure 6 (right) shows these two curves when

\begin{equation}
C_p = \pi - \arcsin \left( -\frac{\omega}{2\kappa \sqrt{1 + \alpha^2}} - \omega \tau - \arctan(\alpha) \right).
\end{equation}

The red curve corresponds to the equation $\text{Re}|A - B|_{\lambda=i\beta} = 0$ while the blue one corresponds to $\text{Im}|A - B|_{\lambda=i\beta} = 0$. One can check that the two equations above for solving $C_p$ are equivalent to (32). In total we have six Hopf bifurcation points coming from the $(A - B)$ block.

5.6. Numerical continuation with DDE-Biftool. In this section, we use DDE-Biftool to determine the branch stability and bifurcation points for given branches of CLM in the case $n = 3$ with both unidirectional and bidirectional coupling and for $n = 8$ in the unidirectional case. We begin with the three-laser system with unidirectional coupling. By Proposition 5.1, we do not expect any pitchfork bifurcation points.

We consider the branch of CLMs with isotropy subgroup $H_{1}^{0}$. Figure 7 (left) shows the stability of $H_{1}^{0}$ together with the bifurcation points. We use (+) for saddle-node bifurcation points and (*) for Hopf bifurcation points. The color scheme to denote branch stability is green for stable and magenta for unstable. The figure also shows a magnification of the part where two bifurcation points are close. We plot, in Figure 7 (right), in $C_p N$ plane the branches of CLMs with isotropy subgroup $H_{1}^{01}$ and $H_{1}^{02}$ isomorphic to $\mathbb{Z}_3(\gamma, \pm 2\pi/3)$. Those branches are
mapped into each other using the 2π/3-translational symmetry (7) and each branch has the same branch stability and bifurcation points as the branch of CLMs $H_{θ}^{0}$, with $H_{1}^{θ_0}$ plotted with a thicker line.

We follow the bifurcating branches of CLMs in DDE-Biftool. From Proposition 5.2, we expect only saddle-node and pitchfork bifurcations from the CLM with isotropy subgroup $D_3(γ, μ)$ and the same holds for the other CLMs. In the branch of CLMs corresponding to the CLM with isotropy subgroup $D_3(γ, μ)$, the pitchfork bifurcation points and saddle-node bifurcation points come from different diagonal blocks and so they can be distinguished. Now consider the branch of CLMs corresponding to $Z_2(μ)$ (see Figure 3, large branch), in this case,
we expect pitchfork bifurcations. Figure 8 (left) shows a symmetry-breaking bifurcation point marked with the symbol (⋄) and the bifurcating branch of solution (dark green) that emanates from this pitchfork bifurcation point. We also verify that CLMs on this bifurcating branch have isotropy subgroup \( \mathbb{Z}_2(\mu) \). From the branch of \( D_3(\gamma, \mu) \) CLMs shown in Figure 8 (right) we follow one of its pitchfork bifurcation points in DDE-Biftool and obtain a bifurcating branch of CLMs (aqua). This branch of CLMs has \( \mathbb{Z}_2(\mu) \) symmetry and is the same as the branch CLMs with isotropy subgroup \( \mathbb{Z}_2(\mu) \) mentioned above.

\[ \begin{align*}
\text{Figure 8.} & \quad \text{Left: Bifurcating branch of CLMs (dark green) emanating from the pitchfork bifurcation point (⋄) on the } \mathbb{Z}_2(\mu) \text{ branch of CLMs. Right: A bifurcating branch of } \mathbb{Z}_2(\mu) \text{ CLM emanating from the pitchfork bifurcation points of the } D_3(\gamma, \mu) \text{ CLM.}
\end{align*} \]

**Bifurcating branches of periodic solutions.** The equivariant Hopf bifurcation theorem classifies the symmetry of these bifurcating branches of periodic solution. We start with the Hopf bifurcation points along the branch of \( H^\theta_1 \) CLMs of the unidirectionally coupled network. We have identified all branches of periodic orbits with identical isotropy subgroups with the same color: see Figure 9 (left). Note that the bold red curve bifurcating from the Hopf point near \( C_p = 11, N = -0.1 \) is the only asymptotically stable branch.

We now proceed with the two other branches of CLMs with isotropy subgroup isomorphic to \( \mathbb{Z}_3(\gamma, \pm 2\pi/3) \) in the unidirectional case. We see in Figure 9 (right) that these two branches have the same stability and bifurcation points as the branch of CLMs with isotropy subgroup \( \mathbb{Z}_3(\gamma) \). Indeed, following the Hopf bifurcation points on these branches results to the same bifurcating branches of periodic solutions as in Figure 9 (left) except that they are shifted by \( 2\pi/3 \) to the left and to the right. Figure 9 (right) shows the three branches of CLMs together with their bifurcating branches of periodic solutions. The three branches of periodic solutions shown in Figure 9 have a more elaborate connection. Here, we see that two of the branches end on the third branch on period-doubling bifurcation points. Figure 10 shows that branches of periodic solutions in the middle of the figure serve as bridges to pairs of Hopf bifurcation points, including transitions via period-doubling branches (\( \Delta \)). Note that two torus bifurcation points on the blue branch have been removed for clarity. The bridge phenomenon is originally observed in [29, 19] in continuation with DDE-Biftool of a model for an optically injected single laser and these authors also find period-doubling branches. In
this case, the bridge solutions connect mode and antimode branches of periodic solutions as the feedback rate is varied. Bridges are also observed for the 2-laser model in [11]. Figure 11 (left) shows the period-doubled branch of periodic solutions which is the top branch from the period-doubling point in Figure 10 and Figure 11 (right) is the period-doubled branch which is the lower branch from the period-doubling point in Figure 10.

We now look at the bifurcating branches of periodic solutions for the three-laser network with bidirectional coupling. The branch corresponding to the $D_3(\gamma, \mu)$ CLM has eight Hopf bifurcation points; see Figure 12 (left). Notice that the green branch which follows a $\mathbb{Z}_2(\mu)$
symmetric branch of periodic solutions. Now, $D_3$ symmetry-breaking Hopf bifurcation leads to three branches of periodic solutions bifurcating simultaneously with symmetry groups $Z_2(\mu)$, $Z_2(\mu, \pi)$, and $Z_3(\gamma, 2\pi/3)$. Figure 12 (right) shows a zoom-in of the $D_3$ branch on a different run of DDE-Biftool where from that same Hopf bifurcation point emerges the $Z_3(\gamma, 2\pi/3)$ branch (between $C_p = -7$ and $C_p = -6$).

In Figure 12 (left), we color the bifurcating branches of periodic solutions based on their symmetry group. In Figure 13, we plot the coordinates $\text{Re}(E_j(t))$, $j = 1, 2, 3$ for each branch color. Branches in blue are in-phase with isotropy subgroups $D_3(\gamma, \mu)$, which means that those two Hopf bifurcations have trivial isotropy subgroup since they are obtained from the diagonal block ($A+2B$). On the other hand, the other six Hopf bifurcation points are obtained from the $A-B$ block and hence these six Hopf bifurcations give symmetry-breaking bifurcating branches of periodic solutions. The symmetry groups of these bifurcating branches are $Z_3(\gamma, -2\pi/3)$ for the five branches in red and $Z_2(\mu)$ for the lone branch in green.

We conclude with Figure 14 which shows a (2$\pi$/8-translationally symmetric) family of grey branches of CLMs with isotropy subgroups isomorphic to $Z_8$ along with a wide range of bridge connections between the branches. The red branches of Hopf bifurcation have isotropy subgroup isomorphic to $Z_2$, the orange branches have isotropy subgroup isomorphic to $Z_4$, and the aqua branches also have isotropy subgroups isomorphic to $Z_8$; they emerge from the $A+7B$ block. Two $Z_4$ (orange) branches connect to a $Z_2$ (red) branch via period-doubling bifurcation ($\triangle$). Both of the two $Z_4$ branches have period approximately 120 while the $Z_2$ branch has period near 60.

6. Discussion and future work. This work uses tools from equivariant bifurcation theory and numerical continuation to partially extend the results of [11] to unidirectional and bidirectional symmetrically coupled rings of Lang–Kobayashi equations modeling semiconductor lasers. The main results of this paper are the following. We use group-theoretic methods to classify CLMs in terms of isotropy subgroup and obtain existence results for certain classes of isotropy subgroups; in particular for maximal isotropy subgroups. Using standard methods
Figure 12. Left: Stability and bifurcation points along the bifurcating branches of periodic solutions. Branches of periodic solutions bifurcating from Hopf points along the branch of $D_3(\gamma,\mu)$ CLMs. Branches with the same color have the same symmetry group. Notice the green branch with $Z_2(\mu)$-symmetry. Right: A zoom-in of a different run of DDE-Biftool now following the $Z_3(\gamma,2\pi/3)$ branch.

Figure 13. Symmetries of the bifurcating branches of periodic solutions. Left: (red) $Z_3(\gamma,-2\pi/3)$; Middle: (blue) $D_3(\gamma,\mu)$; Right: (green) $Z_2(\mu)$.

of equivariant bifurcation theory we study the linearization and bifurcations at CLMs and from this we are able to compute the location of steady-state and Hopf bifurcation points on branches of CLMs and classify the type of symmetry-breaking bifurcation point. This confirms and complements the location of bifurcation points obtained via numerical continuation using DDE-Biftool. Moreover, we perform a short exploration of parameter space by numerical continuation methods for three and eight laser networks. As in previous studies, we also find branches of periodic solutions which act as bridges between various branches of solutions. Our simulations show that those bridge connections between various branches are prevalent and provide for a rich bifurcation structure for unidirectionally coupled rings, while the bridge structure in the bidirectional coupling case is more modest. Further studies of this issue would be needed. As discussed in the introduction, the focus of our results blends well
with recent experimental and mathematical modeling contributions on networks of coupled lasers.

Parameter continuation of bridge connections has been studied in Piéroux et al. [29] for a single optically injected laser. The bridges found by varying $\kappa$ in [19] are perturbed by varying a second parameter: $\alpha$ and the coupling strength, respectively. Such two-parameter analysis should also be performed on coupled LK systems (1). The coupling strength $\kappa$ is certainly one parameter that can be varied. However, note that since the modelization via the coupled LK equations is valid for weak coupling [23], results obtained for large coupling strength should be eventually verified experimentally for relevance of the model as in [17].

The question of the dynamics of networks (1) and (2) under forced symmetry-breaking via the detuning parameter is of importance because it describes a more realistic situation and the results obtained will be robust to small/noise perturbations. Evidence from two $\mathbb{Z}_2$ symmetrically coupled lasers shows that stable synchrony achieved under assumption of identical coupling phase loses stability with added noise perturbations [24]. In particular, one should investigate how the bridge connections are affected by turning on the detuning parameter.

Quasi-periodic solutions are observed in the neighborhood of the bridge solutions [19] and are linked with low-frequency fluctuation (LFF) dynamics. The neighborhood of torus bifurcation points should also be studied to determine the type of quasiperiodic behavior present in the unidirectional network and especially, the projection of the dynamics on each single laser. Continuation of tori for ODEs is proposed by Schilder, Osinca, and Vogt [30]. No such method yet exists for DDEs.

One could also explore other symmetric schemes such as all lasers coupled via a common
mirror \([14, 22]\) leading to \(S_n\) symmetry or more complex topologies (possibly symmetric) such as described in \([1]\) in the context of two-mode lasers. At a more theoretical level, it would be interesting to derive rate equations for networks of \(n\) coupled lasers as in Mulet, Masoller, and Mirasso \([23]\).

The classification of CLMs with respect to isotropy subgroups can be used to make predictions about the existence of sublattice synchronized solutions such as exhibited by Nixon et al. \([25]\). In particular, the coupling scheme developed in \([25]\) not only can enable the testing of CLM solutions studied in this paper, but more complicated network topologies can also be obtained. The stability of the sublattice synchronized CLMs using networks of LK equations would also be of interest. For nonsymmetric network topologies, the coupled cell network theory developed by Golubitsky, Stewart, and collaborators \([15]\) gives sufficient conditions for obtaining patterns of synchrony. In some cases, synchrony patterns can be shown to exist in quotient networks which have symmetry and therefore the results of this paper are relevant.

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**REFERENCES**


