EQUIVARIANT VERSAL UNFOLDINGS FOR LINEAR RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We continue our investigation of versality for parametrized families of linear retarded functional differential equations (RFDEs) projected onto finite-dimensional invariant manifolds. In this paper, we consider RFDEs equivariant with respect to the action of a compact Lie group. In a previous paper (Buono and LeBlanc, J. Diff. Eqs., 193, 307-342 (2003)), we have studied this question in the general case (i.e. no a priori restrictions on the RFDE). When studying the question of versality in the equivariant context, it is natural to want to restrict the range of possible unfoldings to include only those which share the same symmetries as the original RFDE, and so our previous results do not immediately apply. In this paper, we show that with appropriate projections, our previous results on versal unfoldings of linear RFDEs can be adapted to the case of linear equivariant RFDEs. We illustrate our theory by studying the linear equivariant unfoldings at double Hopf bifurcation points in a $D_3$-equivariant network of coupled identical neurons modeled by delay-differential equations due to delays in the internal dynamics and coupling.

1. Introduction. Symmetry plays an important role in the description of many physical systems. Whether the symmetry is an intrinsic part of the physical system, or whether it is merely a modeling assumption, the mathematical model for any such system must reflect this symmetry. When the mathematical model is a differential equation, this is achieved by imposing equivariance of the differential equation. That is, if $\Gamma$ is a group of transformations which represents the symmetries of the physical system, then the differential equation model must commute with an action of $\Gamma$. An important consequence of $\Gamma$-equivariance is that any solution of the differential equation is mapped onto other solutions by the elements of $\Gamma$. Thus, the algebraic structure of the group $\Gamma$ yields important information about the geometrical structure of solutions of the differential equation in phase space.

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The theory of equivariant dynamical systems is by now well-developed and understood (see, for example [8] and [9]). Γ-equivariance imposes strong algebraic restrictions on the functional form of the differential equation. This has important implications, for example, in the study of local stability and bifurcation of equilibrium points. In particular, since the linearization of the vector field at the equilibrium point must commute with Γ, then this linearization usually has eigenvalues of high geometric multiplicity.

In this paper, we will be interested in systems which are modeled by parametrized families of retarded functional differential equations

$$\dot{z}(t) = L(\alpha)z_t + F(z_t, \alpha), \quad \alpha \in \mathbb{C}^p,$$

where \(L(\alpha)\) is a parametrized family of bounded linear functional operators and \((u, \alpha) \mapsto F(u, \alpha)\) is nonlinear and smooth with \(F(0, \alpha) = 0\) and \(D_u F(0, \alpha) = 0\) for all \(\alpha\), such that (1) is equivariant under the action of a compact Lie group of symmetries \(\Gamma\) for all \(\alpha\). Examples of such systems can be found in the study of neural networks modeled by symmetrically coupled cell systems with delays in the coupling ([13, 14, 16, 17]), in the study of semiconductor lasers subject to optical feedback [12, 15] or symmetrically coupled arrays of lasers [11]. Specifically, we are interested in the following question regarding (1): consider the equilibrium \(x = 0, \alpha = 0\) of (1), and let \(V\) be a finite-dimensional invariant manifold for the dynamics of (1) which passes through this equilibrium point. Let

$$\dot{x} = B(\alpha)x + G(x, \alpha)$$

be the parametrized family of ordinary differential equations which gives the dynamics of (1) restricted to \(V\), where \(B(\alpha)\) is a parametrized family of matrices, and \(G\) is the nonlinearity. Under what conditions on \(L(\alpha)\) in (1) can we be assured that \(B(\alpha)\) is a versal unfolding of the matrix \(B(0)\) within the space of \(\Gamma\)-equivariant matrices! Recall that, loosely speaking, the family \(B(\alpha)\) is a versal unfolding of \(B(0)\) if it is, up to similarity transformations and changes of parameters, the most general \(C^\infty\) perturbation of \(B(0)\) (a precise definition is given in [3] in the absence of \(\Gamma\)-equivariance, and in Section 3 of this paper in the \(\Gamma\)-equivariant case). The above question is especially important in the case where (2) represents a center manifold reduction of (1). If \(B(\alpha)\) is not versal, then this could lead to restrictions on the bifurcation behavior of (1) near the equilibrium. It is clear that answering this question does not depend on the nonlinearity \(F\) in (1); therefore, we restrict our attention to the class of parametrized families of \(\Gamma\)-equivariant linear retarded functional differential equations.

This problem was studied in [3] in the absence of \(\Gamma\)-equivariance. In particular, Theorem 7.4 of that paper gives a method for constructing a parametrized family \(L(\alpha)\) of bounded linear functional operators such that the family of matrices \(B(\alpha)\) in (2) is a versal unfolding of the matrix \(B(0)\). Our approach in this paper will be to construct a suitable equivariant projection of the family \(L(\alpha)\) of Theorem 7.4 of [3], and to show that the resulting associated \(\Gamma\)-equivariant family of matrices \(B(\alpha)\) in (2) is a versal unfolding of \(B(0)\) within the space of all \(\Gamma\)-equivariant matrices.

The paper is organized as follows. In Section 2 we review some basic theory of linear retarded functional equations while Section 3 contains the results concerning versal and mini-versal unfoldings of equivariant matrices. In Section 4, our main result about the construction of equivariant unfoldings for linear retarded functional differential equations is presented. Finally, in Section 5 a detailed computation of the equivariant mini-versal unfolding of a \(\mathbb{D}_3\)-equivariant linear delay-differential
equation at a double Hopf bifurcation is discussed in both cases of simple and
double imaginary eigenvalues.

It is assumed that the reader is familiar with the terminology and results of [3],
[5], [6], [7] and [10].

2. Reduction of equivariant linear RFDEs. Let $C_n = C([-\tau, 0], \mathbb{C}^n)$ be the
Banach space of continuous functions from the interval $[-\tau, 0]$, into $\mathbb{C}^n$ ($\tau > 0$)
endowed with the norm of uniform convergence. We will first consider the linear
homogeneous RFDE

$$\dot{z}(t) = L_0(z(t)), \quad (3)$$

where $L_0$ is a bounded linear operator from $C_n$ into $C_n$. We write

$$L_0(\varphi) = \int_{-\tau}^{0} d\eta(\theta)\varphi(\theta),$$

where $\eta$ is an $n \times n$ matrix-valued function of bounded variation defined on $[-\tau, 0]$.

We will suppose that (3) has certain symmetry properties. Specifically, we let $\Gamma$
be a compact group of transformations acting linearly on $C_n$. We say that (3) is
$\Gamma$-equivariant if

$$\gamma \cdot d\eta(\theta) = d\eta(\theta) \cdot \gamma, \quad \forall \gamma \in \Gamma, \theta \in [-\tau, 0]. \quad (4)$$

A consequence of this definition is that if $x(t)$ is a solution to (3), then so is
$\gamma \cdot x(t)$, for all $\gamma \in \Gamma$.

Suppose that $\Lambda \subset \mathbb{C}$ is a non-empty finite set of eigenvalues of the infinitesimal
generator $A_0$ for the semi-flow of (3), i.e. $\lambda \in \Lambda$ if and only if $\lambda$ satisfies the
characteristic equation

$$\det \Delta(\lambda) = 0, \quad \text{where} \quad \Delta(\lambda) = \lambda I_n - \int_{-\tau}^{0} d\eta(\theta)e^{\lambda\theta}, \quad (5)$$

where $I_n$ is the $n \times n$ identity matrix. Using adjoint theory, it is known that we can write

$$C_n = P \oplus Q \quad (6)$$

where the generalized eigenspace $P$ corresponding to $\Lambda$ and the complementary
subspace $Q$ are both invariant under the semiflow of (3), and invariant under $A_0$.

Define $C_n^* = C([0, \tau], \mathbb{C}^n)$, where $\mathbb{C}^n$ is the $n$-dimensional space of row vectors.
We have the adjoint bilinear form on $C_n^* \times C_n$:

$$(\psi, \varphi) = \psi(0)\varphi(0) - \int_{-\tau}^{0} \int_{0}^{\theta} \psi(\xi - \theta)d\eta(\theta)\varphi(\xi)d\xi. \quad (7)$$

We let $\Phi = (\varphi_1, \ldots, \varphi_c)$ be a basis for $P$, and $\Psi = \text{col}(\psi_1, \ldots, \psi_c)$ be a basis for the
dual space $P^*$ in $C_n^*$, chosen so that $(\Psi, \Phi)_n$ is the $c \times c$ identity matrix, $I_c$.

In this case, we have $Q = \{\varphi \in C_n : (\Psi, \varphi)_n = 0\}$. It follows that $\Phi' = \Phi B$
where $B$ is a $c \times c$ constant matrix. The spectrum of $B$ coincides with $\Lambda$. Using the
decomposition (6), any solution to (3) can be written as $z = \Phi x + y$, where
$x \in C^c$ and $y \in Q$ is a $C^1$ function. The dynamics of (3) on $P$ are then given by the
c-dimensional linear ordinary differential equation

$$\dot{x} = Bx. \quad (8)$$

The important point, as we will now show, is that $\Gamma$-equivariance of (3) forces $\Gamma$-
invariance of the splitting (6) and $\Gamma$-equivariance of the reduced ordinary differential equation (8).
Proposition 2.1. The spaces $P$, $Q$ and $P^*$ defined above are $\Gamma$-invariant, and the $c \times c$ matrix $B$ in (8) commutes with a representation $G : \Gamma \rightarrow GL(c, \mathbb{C})$ of the group $\Gamma$.

Proof. We start by noting the following identity which follows trivially from $\Gamma$-equivariance of (3), and from (7):

$$\langle \Psi, \gamma \cdot \Phi \rangle_n = (\Psi \cdot \gamma, \Phi)_n \quad \forall \gamma \in \Gamma. \tag{9}$$

The functions $\varphi_1, \ldots, \varphi_c$ which span $P$ are generalized eigenfunctions of the operator $A$ defined by $A \varphi = \varphi'$ whose domain is

$$\mathcal{D}(A) = \left\{ \varphi \in C_\nu : \varphi' \in C_\nu \text{ and } \varphi'(0) = \int_{-\tau}^{0} d\eta(\theta) \varphi(\theta) \right\}.$$ 

Let $\gamma \in \Gamma$, $\varphi \in P$, and consider $\tilde{\varphi} = \gamma \cdot \varphi$. Then obviously $\tilde{\varphi}' \in C_\nu$ since $\varphi' \in C_\nu$. Moreover, from (4) and the fact that $\varphi \in \mathcal{D}(A)$, we get

$$\tilde{\varphi}'(0) = \gamma \cdot \varphi'(0) = \gamma \int_{-\tau}^{0} d\eta(\theta) \varphi(\theta) = \int_{-\tau}^{0} d\eta(\theta) \gamma \cdot \varphi(\theta) = \int_{-\tau}^{0} d\eta(\theta) \tilde{\varphi}(\theta),$$

so $\tilde{\varphi} \in \mathcal{D}(A)$. Now, there exists an integer $k$ and a $\lambda \in \Lambda$ such that $(A - \lambda I)^k \varphi = 0$. Since $A (\gamma \cdot \varphi) = (\gamma \cdot \varphi)' = \gamma \cdot \varphi' = \gamma \cdot A \varphi$, it is easy to see that $(A - \lambda I)^k \tilde{\varphi} = 0$, and we conclude that $\tilde{\varphi} \in P$. Thus, $P$ is $\Gamma$-invariant. A similar argument shows that $P^*$ is also $\Gamma$-invariant. This implies that for all $\gamma \in \Gamma$, there exist $c \times c$ matrices $G(\gamma)$ and $H(\gamma)$ such that

$$\gamma \cdot \Phi = \Phi \cdot G(\gamma) \quad \text{and} \quad \Psi \cdot \gamma = H(\gamma) \cdot \Psi. \tag{10}$$

In fact, it follows from (9) and the fact that $\langle \Psi, \Phi \rangle_n = I_\nu$ that

$$H(\gamma) = H(\gamma)(\Psi, \Phi)_n = (H(\gamma) \cdot \Psi, \Phi)_n = (\Psi \cdot \gamma, \Phi)_n = (\Psi, \gamma \cdot \Phi)_n =

(\Psi, \Phi \cdot G(\gamma))_n = (\Psi, \Phi)_n G(\gamma) = G(\gamma).$$

Note that

$$G(\sigma \gamma) = (\Psi, \Phi)_n G(\sigma \gamma) = (\Psi, \Phi \cdot G(\sigma \gamma))_n = (\Psi, \sigma \gamma \cdot \Phi)_n =

(\Psi \cdot \sigma, \gamma \cdot \Phi)_n = (\Psi \cdot \sigma \cdot \gamma \cdot \Phi)_n = G(\sigma \gamma) G(\gamma) = G(\sigma) G(\gamma),$$

so that the mapping

$$\Gamma \rightarrow GL(c, \mathbb{C}) \quad \gamma \mapsto G(\gamma)$$

is a representation of the group $\Gamma$ in the space of invertible $c \times c$ matrices.

Now, $\varphi \in Q$ if and only if $\langle \Psi, \varphi \rangle_n = 0$. Then using (9), one gets that

$$\langle \Psi, \gamma \cdot \varphi \rangle_n = (\Psi \cdot \gamma, \varphi)_n = (G(\gamma) \cdot \Psi, \varphi)_n = G(\gamma)(\Psi, \varphi)_n = 0,$$

so $Q$ is $\Gamma$-invariant.

Finally, it follows from the fact that $B = (\Psi, \Phi')_n$ that

$$B \cdot G(\gamma) = G(\gamma) \cdot B \quad \forall \gamma \in \Gamma. \tag{11}$$

□
3. Equivariant versal unfoldings of matrices. Let $B \in \text{Mat}_{c \times c}$, where $\text{Mat}_{c \times c}$ denotes the space of $c \times c$ matrices with entries in $\mathbb{C}$. Recall that a $p$-parameter unfolding of $B$ is a $C^\infty$ mapping $\varphi : \mathbb{C}^p \rightarrow \text{Mat}_{c \times c}$ such that $\varphi(\alpha) = B$ for some $\alpha \in \mathbb{C}^p$. Then we say that a $p$-parameter unfolding $\varphi(\alpha)$ of $B$ is a versal unfolding of $B$ if, given any $q$-parameter unfolding $A(\beta)$ of $B$ (with $A(\beta) = B$), there exists a $C^\infty$ mapping

$$\phi : \mathbb{C}^q \rightarrow \mathbb{C}^p,$$

with $\phi(\beta_0) = \alpha_0$, and a $q$-parameter unfolding $C(\beta)$ of the identity matrix such that

$$A(\beta) = C(\beta)\varphi(\phi(\beta))(C(\beta))^{-1}, \quad \forall \beta \text{ near } \beta_0.$$

If $B$ is a versal unfolding which depends on the least number of parameters, then $B$ is called a mini-versal unfolding. Thus, one may view a versal unfolding of $B$ as the most general $C^\infty$ perturbation of $B$ up to similarity and change of parameters. One then has the following sufficient criterion [1] for establishing the versality of a given unfolding of $B$:

**Proposition 3.1.** Let $B(\alpha)$ be a $p$-parameter unfolding of $B \in \text{Mat}_{c \times c}$. Let $\Sigma$ denote the similarity orbit of $B$, i.e.,

$$\Sigma = \{ CBC^{-1} : C \in GL(c, \mathbb{C}) \} \subset \text{Mat}_{c \times c}.$$

If

$$\text{Mat}_{c \times c} = T_B \Sigma + D_{\alpha} B(\alpha_0) : \mathbb{C}^p,$$

(12)

where $T_B \Sigma$ is the tangent space to $\Sigma$ at $B$, then $B$ is a versal unfolding of $B$. If, in addition, the codimension of $T_B \Sigma$ in $\text{Mat}_{c \times c}$ is equal to $p$, then $B$ is mini-versal.

In this paper, we are not interested in the most general perturbation of $B$, since we are only considering matrices in $\text{Mat}_{c \times c}$ which commute with the group $\Gamma$. In fact, we need only consider perturbations in the subspace $\text{Mat}^\Gamma_{c \times c}$ defined by

$$\text{Mat}^\Gamma_{c \times c} = \{ A \in \text{Mat}_{c \times c} : G(\gamma)A = AG(\gamma), \quad \forall \gamma \in \Gamma \},$$

where $G$ is the representation of $\Gamma$ in $GL(c, \mathbb{C})$ defined in (10). We therefore adopt the following definition:

**Definition 3.2.** A $p$-parameter unfolding $B(\alpha)$ of $B \in \text{Mat}^\Gamma_{c \times c}$ is said to be a $\Gamma$-unfolding of $B$ if $B(\alpha) \in \text{Mat}^\Gamma_{c \times c}$ for all $\alpha$. The $\Gamma$-unfolding is said to be $\Gamma$-versal if, given any $q$-parameter $\Gamma$-unfolding $A(\beta)$ of $B$ (with $A(\beta) = B$), there exists a $C^\infty$ mapping

$$\phi : \mathbb{C}^q \rightarrow \mathbb{C}^p,$$

with $\phi(\beta_0) = \alpha_0$, and a $q$-parameter $\Gamma$-unfolding $C(\beta)$ of the identity matrix such that

$$A(\beta) = C(\beta)\varphi(\phi(\beta))(C(\beta))^{-1}, \quad \forall \beta \text{ near } \beta_0.$$

If $B$ is a $\Gamma$-versal unfolding which depends on the least number of parameters, then $B$ is called a $\Gamma$-mini-versal unfolding.

The following lemma, which is the $\Gamma$-equivariant version of Proposition 3.1, is proved in Appendix A.

**Lemma 3.3.** Let $B(\alpha)$ be a $p$-parameter $\Gamma$-unfolding of $B \in \text{Mat}^\Gamma_{c \times c}$. Let $\Sigma^\Gamma$ denote the $\Gamma$-similarity orbit of $B$, i.e.,

$$\Sigma^\Gamma = \{ CBC^{-1} : C \in GL(c, \mathbb{C}) \cap \text{Mat}^\Gamma_{c \times c} \} \subset \text{Mat}^\Gamma_{c \times c}.$$
If

\[ \operatorname{Mat}^\Gamma_{\times c} = T_B \Sigma^\Gamma + D_\alpha B(\alpha_0) \cdot \mathbb{C}^p, \]  

(13)

where \( T_B \Sigma^\Gamma \) is the tangent space to \( \Sigma^\Gamma \) at \( B \), then \( B \) is a \( \Gamma \)-versal unfolding of \( B \). If, in addition, the codimension of \( T_B \Sigma^\Gamma \) in \( \operatorname{Mat}^\Gamma_{\times c} \) is equal to \( p \), then \( B \) is \( \Gamma \)-mini-versal.

**Lemma 3.4.** There exists a projection \( \pi^\Gamma \circlearrowleft: \operatorname{Mat}^\Gamma_{\times c} \longrightarrow \operatorname{Mat}^\Gamma_{\times c} \) such that

\[ \pi^\Gamma \circlearrowleft(AM) = A \pi^\Gamma \circlearrowleft(M) \quad \text{and} \quad \pi^\Gamma \circlearrowleft(MA) = \pi^\Gamma \circlearrowleft(M) A, \quad \forall A \in \operatorname{Mat}^\Gamma_{\times c}, \; M \in \operatorname{Mat}^\Gamma_{\times c}. \]  

(14)

**Proof.** For \( M \in \operatorname{Mat}^\Gamma_{\times c} \), define

\[ \pi^\Gamma \circlearrowleft(M) = \int G(\gamma)MG(\gamma^{-1}) \, d\gamma \]

where the integral denotes the Haar integral on \( \Gamma \), which we assume has been normalized so that the measure of \( \Gamma \) is 1 (recall that we assume that \( \Gamma \) is compact). Since \( G \) is a morphism and since the Haar integral is translation invariant, we have that for all \( \sigma \in \Gamma \)

\[ G(\sigma) \pi^\Gamma \circlearrowleft(M)(G(\sigma))^{-1} = \int G(\sigma\gamma)MG((\sigma\gamma)^{-1}) \, d\gamma = \pi^\Gamma \circlearrowleft(M), \]

so that \( \pi^\Gamma \circlearrowleft(M) \in \operatorname{Mat}^\Gamma_{\times c} \). Moreover, we clearly have that \( \pi^\Gamma \circlearrowleft(M) = M \) if and only if \( M \in \operatorname{Mat}^\Gamma_{\times c} \). We then conclude that \( \pi^\Gamma \circlearrowleft \) is idempotent, with range \( \operatorname{Mat}^\Gamma_{\times c} \). Finally, (14) follows from the definition of \( \pi^\Gamma \circlearrowleft \) and the fact that \( A \) commutes with \( G(\gamma) \), for all \( \gamma \in \Gamma \).

**Proposition 3.5.** Let \( B \) be a \( p \)-parameter unfolding of \( B \in \operatorname{Mat}^\Gamma_{\times c} \) such that (12) holds (which implies by Proposition 3.1 that \( B \) is a \( \Gamma \)-versal unfolding of \( B \)). Let \( \pi^\Gamma \circlearrowleft \) be the projection of \( \operatorname{Mat}^\Gamma_{\times c} \) onto \( \operatorname{Mat}^\Gamma_{\times c} \) such as in Lemma 3.4. Then \( B^\Gamma \equiv \pi^\Gamma \circlearrowleft(B) \) is a \( \Gamma \)-versal unfolding of \( B \).

**Proof.** We first show that \( \pi^\Gamma \circlearrowleft(T\Sigma_B) = T\Sigma_B \). Let \( x = [B, y] \in T\Sigma_B \), where \( y \in \operatorname{Mat}^\Gamma_{\times c} \). Since obviously \( x \in \operatorname{Mat}^\Gamma_{\times c} \), then \( x = \pi^\Gamma \circlearrowleft(x) = \pi^\Gamma \circlearrowleft([B, y]) \in \pi^\Gamma \circlearrowleft(T\Sigma_B) \), since \([B, y]\) is also an element of \( T\Sigma_B \). Thus \( T\Sigma_B \subset \pi^\Gamma \circlearrowleft(T\Sigma_B) \). Now let \( x = \pi^\Gamma \circlearrowleft(T\Sigma_B) \), i.e. \( x = \pi^\Gamma \circlearrowleft(By - yB) \) for some \( y \in \operatorname{Mat}^\Gamma_{\times c} \). Then, from Lemma 3.4, we have

\[ x = \pi^\Gamma \circlearrowleft(By - yB) = \pi^\Gamma \circlearrowleft(By) - \pi^\Gamma \circlearrowleft(yB) = [B, \pi^\Gamma \circlearrowleft(y)] \in T\Sigma_B. \]

Therefore, we have shown that \( \pi^\Gamma \circlearrowleft(T\Sigma_B) = T\Sigma_B \).

Now, we will show that \( \pi^\Gamma \circlearrowleft(D_\alpha B(\alpha_0) \cdot \mathbb{C}^p) = D_\alpha B^\Gamma(\alpha_0) \cdot \mathbb{C}^p \). We have that \( x \in D_\alpha B^\Gamma(\alpha_0) \cdot \mathbb{C}^p \), if and only if there exists \( v \in \mathbb{C}^p \) such that

\[ x = \left. \frac{d}{de} \left( B^\Gamma(\alpha_0 + \varepsilon v) \right) \right|_{e=0} = \left. \frac{d}{de} \left( \pi^\Gamma \circlearrowleft(B(\alpha_0 + \varepsilon v)) \right) \right|_{e=0} = \pi^\Gamma \circlearrowleft \left( \left. \frac{d}{de} \left( B(\alpha_0 + \varepsilon v) \right) \right|_{e=0} \right) \in \pi^\Gamma \circlearrowleft(D_\alpha B(\alpha_0) \cdot \mathbb{C}^p). \]

Therefore, \( \pi^\Gamma \circlearrowleft(D_\alpha B(\alpha_0) \cdot \mathbb{C}^p) = D_\alpha B^\Gamma(\alpha_0) \cdot \mathbb{C}^p \).

Finally, we show that

\[ \operatorname{Mat}^\Gamma_{\times c} = T\Sigma_B + D_\alpha B^\Gamma(\alpha_0) \cdot \mathbb{C}^p. \]
Obviously, from the preceding results, we have that
\[ TΣ_B + D_αB^Γ(α_0) \cdot C^p \subset Mat^Γ_{e\times e}. \]

Let \( x \in Mat^Γ_{e\times e} \), then from (12) we have that \( x = x_a + x_b \), where \( x_a \in TΣ_B \) and \( x_b \in D_αB(α_0) \cdot C^p \). Therefore
\[ x = \pi_e^Γ(x) = \pi_e^Γ(x_a) + \pi_e^Γ(x_b) \in \pi_e^Γ(TΣ_B) + \pi_e^Γ(D_αB(α_0) \cdot C^p) = TΣ_B + D_αB^Γ(α_0) \cdot C^p. \]

Thus, we have shown that \( Mat^Γ_{e\times e} = TΣ_B + D_αB^Γ(α_0) \cdot C^p \), and by Lemma 3.3, we conclude that \( B^Γ \) is a \( Γ \)-versal unfolding of \( B \).

**Remark 3.6.** Let \( B \in Mat^Γ_{e\times e} \) be given, and let \( B(α) \) be a \( p \)-parameter unfolding of \( B \) which is mini-versal. Then (12) holds as a direct sum, and by the previous Proposition, \( B^Γ(α) = \pi_e^Γ(B(α)) \) is a \( Γ \)-versal unfolding of \( B \). It should be noted that in general, \( B^Γ(α) \) need not be \( Γ \)-mini-versal, even though \( B \) is mini-versal (in fact it is quite easy to construct examples to illustrate this fact). However, it is always possible to extract a \( Γ \)-mini-versal unfolding of \( B \) from \( B^Γ \), using a straightforward procedure as we now illustrate.

Let \( W \) be any direct sum complement of \( T_BΣ \) in \( Mat_{e\times e} \), i.e.
\[ Mat_{e\times e} = T_BΣ \oplus W. \]  
(15)

For example, if \( W \) is chosen to be the space which is spanned by the matrices \( Ω_1, \ldots, Ω_e \) described in section 6 of [3], then the decomposition of any \( e \times e \) matrix using (15) is particularly simple to compute. Now, for each \( i = 1, \ldots, p \), we define
\[ \hat{B}_i^Γ = \frac{d}{dε} (B^Γ(α_0 + εe_i)) |_{ε=0}, \]
where \( e_i \) is the \( p \)-dimensional vector whose \( i \)-th component is 1 and all other components are zero. Note that
\[ D_αB^Γ(α_0) \cdot C^p = C \cdot \{ \hat{B}_1^Γ, \ldots, \hat{B}_p^Γ \}. \]

Now, decompose these matrices using (15), i.e. write
\[ \hat{B}_i^Γ = [B, y_i] + \sum_{j=1}^{p} θ_{ij}Ω_j, \quad i = 1, \ldots, p, \]
(16)
where \( y_i \in Mat_{e\times e} \), and the \( θ_{ij} \) are scalars. Let \( Θ = (θ_{ij}) \) be the \( p \times p \) matrix coming from the decomposition (16). If we then let rows \( i_1, \ldots, i_k \) of the matrix \( Θ \) determine a maximal set of linearly independent vectors in the rowspace of the matrix \( Θ \), it follows from simple linear algebra that
\[ Mat^Γ_{e\times e} = T_BΣ^Γ \oplus C \cdot \{ \hat{B}_{i_1}^Γ, \ldots, \hat{B}_{i_k}^Γ \}, \]
and so the following \( k \)-parameter \( Γ \)-unfolding of \( B \) is \( Γ \)-mini-versal:
\[ B^Γ(β_1, \ldots, β_k) \equiv B^Γ(α_0 + \sum_{ℓ=1}^{k} β_ℓ e_{i_ℓ}). \]

Recall that the RFDE (3) is defined on \( C^e \), and that \( Γ \) is a subgroup of \( GL(n, C) \). It will be useful to define a projection analogous to \( π_e^Γ \) on \( Mat_{n \times n} \).
Definition 3.7. Let $A \in \text{Mat}_{n \times n}$. We define
\[ \pi^n_\Gamma(A) = \int_\Gamma \gamma A \gamma^{-1} d\gamma. \]

Obviously, it follows that for all $A \in \text{Mat}_{n \times n}$, we have $\sigma \pi^n_\Gamma(A) = \pi^n_\Gamma(A) \sigma$ for all $\sigma \in \Gamma$.

Proposition 3.8. Suppose $C \in \text{Mat}_{c \times c}$ is such that
\[ C = \Psi(0)A\Phi(\tau^*), \]
where $A$ is some $n \times n$ matrix, and $\tau^*$ is some fixed number in $[-\tau, 0]$. Then
\[ \pi^c_\Gamma(C) = \Psi(0)\pi^n_\Gamma(A)\Phi(\tau^*). \]

Proof. This is just a simple computation, using (10). \qed

4. Construction of a $\Gamma$-equivariant $\Lambda$-versal unfolding. Consider now a $C^\infty$ parametrized family of linear RFDEs of the form
\[ \dot{z}(t) = \mathcal{L}(\alpha)(z_t), \]
where $\alpha \in \mathbb{C}^p$, and $\mathcal{L}(\alpha_0) = \mathcal{L}_0$ is as in (3), for some $\alpha_0 \in \mathbb{C}^p$. In the sequel, we will assume that a translation has been performed in the parameter space $\mathbb{C}^p$ such that $\alpha_0 = 0$. It was shown in [3] that for a given set $\Lambda$ of solutions to (5), a parameter-dependent reduction of (18) to the $c$-dimensional generalized eigenspace corresponding to $\Lambda$ is given by
\[ \dot{x} = \tilde{B}(\alpha)x, \]
where
\[ \tilde{B}(\alpha) = B + \Psi(0)[\mathcal{L}(\alpha) - \mathcal{L}_0](\Phi + h(\alpha)), \]
where $h$ is a smooth $n \times c$ matrix-valued function such that $h(0) = 0$, and $B$ is as in (8).

Remark 4.1. The function $h$ is the unique solution (via the implicit function theorem) of a certain nonlinear equation $N(\alpha, h) = 0$ (see equation (4.5) in Proposition 4.1 of [3]). It is a straightforward computation that for all $h$ and all $\gamma \in \Gamma$, we have
\[ N(\alpha, \gamma h G(\gamma^{-1})) = \gamma N(\alpha, h) G(\gamma^{-1}) \]
so that if $h(\alpha)$ is such that $N(\alpha, h(\alpha)) = 0$, then by uniqueness of the solution, we must have $\gamma h(\alpha) G(\gamma^{-1}) = h(\alpha)$, i.e.
\[ \gamma h(\alpha) = h(\alpha) G(\gamma^{-1}) \]
for all $\alpha$ near 0 and for all $\gamma \in \Gamma$. Thus, if (18) is $\Gamma$-equivariant for all $\alpha$, then it follows from (20) that $\tilde{B}(\alpha) \in \text{Mat}_{c \times c}$ for all $\alpha$ near 0 in $\mathbb{C}^p$, i.e. $\tilde{B}$ is a $\Gamma$-unfolding of $B$.

We recall the following definition from [3].

Definition 4.2. The parametrized family of RFDEs (18) is said to be a $\Lambda$-versal unfolding (respectively a $\Lambda$-mini-versal unfolding) for the RFDE (3) if the matrix $\tilde{B}(\alpha)$ defined by (20) is a versal unfolding (respectively a mini-versal unfolding) for $B$.

In the case where (3) is $\Gamma$-equivariant, we have
Definition 4.3. The parametrized family (18) is said to be a $\Gamma$-equivariant $\Lambda$-versal unfolding (respectively a $\Gamma$-equivariant $\Lambda$-mini-versal unfolding) for (3) if (18) is $\Gamma$-equivariant for all $\alpha$, and the matrix $\tilde{B}(\alpha)$ defined by (20) is a $\Gamma$-versal unfolding (respectively a $\Gamma$-mini-versal unfolding) for $B$.

In [3], we showed that given $B \in \text{Mat}_{c \times c}$, there exists points $\tau_0, \tau_1, \ldots, \tau_{c-q}$ in $[-\tau, 0]$, and $n \times n$ matrices $A_j^m$, $j = 0, \ldots, c - q$, $m = 0, \ldots, \delta$ (where $\delta$ is the codimension of $T\Sigma B$ in $\text{Mat}_{c \times c}$) such that the following $\delta$-parameter unfolding of $B$

$$B(\alpha) = B + \sum_{m=1}^{\delta} \alpha_m \left( \sum_{j=0}^{c-q} \Psi(0) A_j^m \Phi(\tau_j) \right)$$

(21)

satisfies (12) as a direct sum (i.e. $B$ is a mini-versal unfolding of $B$). From this, it follows (see Theorem 7.4 of [3]) that if $L(\alpha)$ is the $\delta$-parameter family of bounded linear operators from $C([-\tau, 0], \mathbb{C}^n)$ into $\mathbb{C}^n$ defined in the following way:

$$L(\alpha) z = L_0 z + \sum_{m=1}^{\delta} \alpha_m \left( \sum_{j=0}^{c-q} \pi_{\Lambda}^m A_j^m z(\tau_j) \right),$$

where the $\alpha_m$ are complex parameters and $L_0$ is as in (3), then (18) is a $\Lambda$-versal unfolding of (3).

We now state and prove the main result of this paper.

Theorem 4.4. Suppose $B \in \text{Mat}_{c \times c}^\Gamma$, and consider the matrices $A_j^m$ which appear in the versal unfolding (21) of $B$. Then the family $L(\alpha)$ of $\Gamma$-equivariant bounded linear operators defined by

$$L(\alpha) z = L_0 z + \sum_{m=1}^{\delta} \alpha_m \left( \sum_{j=0}^{c-q} \pi_{\Lambda}^m A_j^m z(\tau_j) \right),$$

(22)

(where $\pi_{\Lambda}^m$ is as in Definition 3.7) is such that (18) is a $\Gamma$-equivariant $\Lambda$-versal unfolding of (3).

Proof. From Propositions 3.5 and 3.8, we conclude that the versal unfolding (21) of $B$ is such that

$$\pi_c^\Gamma(B(\alpha)) = B + \sum_{m=1}^{\delta} \alpha_m \left( \sum_{j=0}^{c-q} \Psi(0) \pi_c^\Gamma(A_j^m) \Phi(\tau_j) \right)$$

is a $\Gamma$-versal unfolding of $B$. Thus, if we define $L(\alpha)$ as in (22), then $\tilde{B}(\alpha)$ (as in (20)) is such that

$$\tilde{B}(\alpha) = \pi_c^\Gamma(B(\alpha)) + \Psi(0)[L(\alpha) - L_0](h(\alpha)).$$

We conclude that $\tilde{B}(\alpha)$ is also a $\Gamma$-versal unfolding of $B$, since $h(0) = 0$ implies that $D_\alpha \tilde{B}(0) = D_\alpha \pi_c^\Gamma(B(\alpha))|_{\alpha=0}$. 

5. An example with $\Gamma = D_3$. In this section we compute the versal unfolding of a $D_3$-equivariant system of linear delay-differential equations near points of double
Hopf bifurcations. Consider the group $\mathbb{D}_3$ generated by \{\kappa, \gamma\} with $\kappa^2 = \gamma^3 = 1$ and the representation $\rho : \mathbb{D}_3 \to GL(3, \mathbb{C})$ given by

$$
\rho(\kappa) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, \quad \rho(\gamma) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}.
$$

(23)

Wu et al. [17] and Neube et al. [14] consider the $\mathbb{D}_3$-equivariant linear delay-differential equation

$$
\dot{u}_j(t) = -u_j(t) + \alpha u_j(t - \tau_n) + \beta [u_{j-1}(t - \tau_n) + u_{j+1}(t - \tau_n)], \quad j = 1, 2, 3
$$

(24)

where the indices are taken mod 3. This linear system appears as the linearization at an equilibrium of a coupled system of three multiple-delayed identical neurons. In those papers, the focus is on finding periodic solutions via the Hopf bifurcation theorem.

The spectrum of the linear operator associated to (24) is obtained by solving the characteristic equation

$$
S(\lambda) = \Delta_1(\lambda)\Delta_2(\lambda)^2 = 0,
$$

where

$$
\Delta_1(\lambda) = \lambda + 1 - \alpha e^{-\lambda \tau_s} - 2\beta e^{-\lambda \tau_n}, \quad \Delta_2(\lambda) = \lambda + 1 - \alpha e^{-\lambda \tau_s} + \beta e^{-\lambda \tau_n}.
$$

Proving analytically the existence of double Hopf bifurcation points in this case is a difficult problem which is beyond the scope of this paper. A common method for assessing the existence of double Hopf bifurcation points is to let $\lambda = i\omega$ and plotting Hopf bifurcation curves as a function of the frequency $\omega$, see [2, 4]. We plot such graphs in Figure 1 for the factors $\Delta_1$ and $\Delta_2$ of the characteristic equation. Double Hopf bifurcation points could also appear as solutions of $\Delta_1(i\omega_1) = \Delta_2(i\omega_2) = 0$. However, in this paper, we are mostly interested in illustrating the $\Gamma$-equivariant unfolding theory rather than doing a complete study of the possible double Hopf bifurcation points in $S(\lambda)$.

Suppose that we are looking for a point of double Hopf bifurcation solution to $\Delta_1 = 0$, then the eigenvalues $\pm i\omega$ must satisfy

$$
\omega + \alpha \sin(\omega \tau_s) = -2\beta \sin(\omega \tau_n) \quad \text{and} \quad 1 - \alpha \cos(\omega \tau_s) = 2\beta \cos(\omega \tau_n).
$$

It is easy to solve these equations for $\alpha$ and $\tau_s$ in terms of the other parameters, we obtain the following two equations

$$
\alpha = \pm \sqrt{(1 - 2\beta \cos(\omega \tau_n))^2 + (\omega + 2\beta \sin(\omega \tau_n))^2}
$$

$$
\tau_s = \frac{1}{\omega} \arctan \left( \frac{-\omega - 2\beta \sin(\omega \tau_n)}{1 - 2\beta \cos(\omega \tau_n)} \right).
$$

Then $\alpha$ and $\tau_s$ with $\beta$ and $\tau_n$ fixed represent curves parametrized by $\omega$ in $(\alpha, \tau_s)$-space. Note that there are several branches of curves depending on the branch of the arctan which is chosen. Figure 1(a) shows some of these curves for $\beta = -0.5$ and $\tau_n = 4$. Points of double Hopf bifurcation lie at the intersection of these curves. Points of double Hopf bifurcation for $\Delta_2 = 0$ are found in a similar way by isolating $\alpha$ and $\tau_s$ as functions of $\beta$, $\tau_n$ and $\omega$. Figure 1(b) shows several curves of imaginary eigenvalues plotted for $\beta = 0.5$ and $\tau_n = 3$.

From the graphics in Figure 1, it is reasonable to suppose that the characteristic equation has a point of double Hopf bifurcation at \((\alpha^*, \beta^*, \tau_s^*, \tau_n^*)\) with $\tau_s^*, \tau_n^*$ nonzero, $\tau_s^* \neq \tau_n^*$ and frequencies $\omega_1 \neq \omega_2$. Of course, the point \((\alpha^*, \beta^*, \tau_s^*, \tau_n^*)\) as
Figure 1. Curves of imaginary eigenvalues in $(\alpha, \tau_s)$-space for (a) $\Delta_1 = 0$ where $\beta = -0.5$ and $\tau_n = 4$, and (b) $\Delta_2 = 0$ where $\beta = 0.5$ and $\tau_n = 3$.

well as the frequencies are different whether $\Delta_1 = 0$ or $\Delta_2 = 0$, however, to keep the notation to a minimum we use this unique label for both cases.

We let $\Lambda = \{ \pm i\omega_1, \pm i\omega_2 \}$ with $\omega_1 \neq 0$ and $\omega_2 \neq 0$ be the set of eigenvalues on the imaginary axis at a double Hopf bifurcation point. If the double Hopf point occurs for the $\Delta_1$ factor, then the eigenvalues of $\Lambda$ are simple and the restriction of the linear operator to the center eigenspace is given by

$$B = \text{diag}(\omega_1i, -\omega_1i, \omega_2i, -\omega_2i).$$

If the double Hopf point occurs for the $\Delta_2$ factor, then the eigenvalues are double and the linear flow on the center eigenspace is given by the following matrix

$$B = \text{diag}(\omega_1i, \omega_1i, -\omega_1i, -\omega_1i, \omega_2i, \omega_2i, -\omega_2i, -\omega_2i).$$

We now compute the $\mathbb{D}_3$-equivariant $\Lambda$-mini-versal unfolding of the linear bounded operator in both cases. Recall that our goal is to find matrices $A_{1m}$ from which we can write (22). The computations rely on the results in [3], mainly Section 6 and Lemma 7.1.

5.1. Simple imaginary eigenvalues. The center eigenspace $P$ has basis

$$\Phi(\theta) = (\phi(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta))$$

$$= (ue^{i\omega_1\theta}, u^e^{-i\omega_1\theta}, u^e^{i\omega_2\theta}, u^e^{-i\omega_2\theta})$$

where $u = (1, 1, 1)^t$. Its dual, $P^*$, has basis $\Psi(s) = (\psi_1(s), \psi_2(s), \psi_3(s), \psi_4(s))^t$ where $\psi_2 = \overline{\psi}_1$, $\psi_4 = \overline{\psi}_3$ and $\psi_j(s) = (\psi_{j1}(s), \psi_{j2}(s), \psi_{j3}(s), \psi_{j4}(s))$, $j = 1, 2, 3, 4$.

Basis of the unfolding space. Let $W$ be the unfolding space spanned by matrices $\Omega_1, \ldots, \Omega_r$ defined in Section 6 of [3]. The dimension of $W$ for matrix $B$ is given by

$$\sum_{j=1}^r \sum_{\ell=1}^{k_j} (2\ell - 1)n_{j, \ell}$$

where $r = 4$ is the number of distinct eigenvalues, $k_j = 1$ is the number of Jordan blocks for each eigenvalue and $n_{j, \ell} = 1$ is the size of each Jordan block, thus the unfolding space is 4-dimensional. The space $\mathcal{E}(W)$ lying in the range of $\Psi(0)$ is linearly isomorphic to $W$ and is also an unfolding space, see Section 6 of [3].
basis of $\mathcal{E}(W)$ is given by $\Psi(0)R^j$ for $j = 1, 2, 3, 4$ where $R^1$, $R^2$, $R^3$, $R^4$ are $3 \times 4$ matrices defined following the procedure established in [3], Section 6.3. In [3] we introduce projections $\Pi_j : \mathbb{C}^3 \to \mathbb{C}^j (j = 1, 2, 3, 4)$ with

$$\Pi_j(v) = \psi_j(0)v^1 + \psi_j(0)v^2 + \psi_j(0)v^3.$$ 

Let $v_j = (v_j^1, v_j^2, v_j^3)^t \in \mathbb{C}^3$ be such that $\Pi_j(v_j) = 1$ and set $R^1 = (v_1, 0, 0, 0)$, $R^2 = (0, v_2, 0, 0)$, $R^3 = (0, v_3, 0)$, and $R^4 = (0, 0, 0, v_4)$. Note that since $\psi_2 = \psi_1$ and $\psi_4 = \psi_3$, we choose $v_2 = \bar{\psi}_1$ and $v_4 = \bar{\psi}_3$.

Since rank $\Phi(0) = 1$, by Theorem 7.2 of [3] we need 4 distinct delays $\tau_0, \tau_1, \tau_2, \tau_3$ to find $3 \times 3$ matrices $A^j_3$ solving

$$R^j = \sum_{j=0}^{3} A^j_3 \Phi(\tau_j).$$

**Projection of the basis.** By Theorem 4.4, the solutions to (26) are used to write the equivariant $A$-unfolding by projecting the matrices $A^m_j$ with $\pi^j_n$. Multiplying (26) by $\Psi(0)$ we obtain equation (17) with $C = \Psi(0)R$. Then, projecting $C$ with $\pi^j_n$, we obtain

$$\pi^j_n(C) = \int_\Gamma G(\gamma)\Psi(0)RG(\gamma^{-1})d\gamma = \int_\Gamma \Psi(0) \cdot \rho(\gamma)RG(\gamma^{-1})d\gamma = \Psi(0) \int_\Gamma \rho(\gamma)RG(\gamma^{-1})d\gamma.$$ 

We project the matrices $R^j$ on the subspace of $\mathbb{D}_3$-equivariant $3 \times 4$ matrices using the projection operator obtained from (27); that is, since the $G$ representation is trivial on $P$ and $\Gamma = \mathbb{D}_3$ is finite, then the projected matrices are

$$\overline{R}^j = \frac{1}{6} \sum_{g \in \mathbb{D}_3} \rho(g)R^j.$$ 

Note that the factor $\frac{1}{6}$ is not necessary to the projection of the matrices and could be removed without further consequences. Thus, we obtain new matrices

$$\overline{R}^1 = \begin{pmatrix} \eta_1 & 0 & 0 & 0 \\ \eta_2 & 0 & 0 & 0 \end{pmatrix}, \quad \overline{R}^2 = \begin{pmatrix} 0 & \eta_2 & 0 & 0 \\ 0 & \eta_2 & 0 & 0 \end{pmatrix},$$

$$\overline{R}^3 = \begin{pmatrix} 0 & 0 & \eta_3 & 0 \\ 0 & 0 & \eta_3 & 0 \end{pmatrix}, \quad \overline{R}^4 = \begin{pmatrix} 0 & 0 & 0 & \eta_4 \\ 0 & 0 & 0 & \eta_4 \end{pmatrix},$$

where $\eta_j = \frac{1}{2}(v_j^1 + v_j^2 + v_j^3)$ and of course $\eta_2 = \bar{\psi}_1$ and $\eta_4 = \bar{\psi}_3$. Because the matrices $\overline{R}^j$ are $\mathbb{D}_3$-equivariant matrices, we may take matrices $A^m_j \mathbb{D}_3$-equivariant with respect to the action of $\mathbb{D}_3$ on $\mathbb{C}^3$ given by (23); that is,

$$A^m_j = \begin{pmatrix} a^{j^m}_1 & b^{j^m}_1 & b^{j^m}_2 \\ b^{j^m}_1 & a^{j^m}_2 & b^{j^m}_3 \\ b^{j^m}_2 & b^{j^m}_3 & a^{j^m}_3 \end{pmatrix}.$$ 

**$\mathbb{D}_3$-mini-versal unfolding** Before solving equation (26), we verify which of the four matrices $\overline{R}^j$ ($j = 1, 2, 3, 4$) are necessary to obtain a mini-versal unfolding. We follow the steps described after Remark 3.6. Since $\mathcal{E}(W)$ is an unfolding space for $B$, we set $\overline{B}^j = \Psi(0)\overline{R}^j$ for $j = 1, 2, 3, 4$. The space $W$ is spanned by $4 \times 4$ matrices
\( \Omega_i \) \((i = 1, 2, 3, 4)\) where \( \Omega_i \) has a 1 on the diagonal at \((i, i)\) and is zero elsewhere. Then, it is easy to compute that
\[
B_{1}^{D_3} = [B, y_1] + \eta_1(\psi_{11}(0) + \psi_{12}(0) + \psi_{13}(0)) \Omega_1
\]
\[
B_{2}^{D_3} = [B, y_2] + \eta_2(\psi_{11}(0) + \psi_{12}(0) + \psi_{13}(0)) \Omega_2
\]
\[
B_{3}^{D_3} = [B, y_3] + \eta_3(\psi_{11}(0) + \psi_{12}(0) + \psi_{13}(0)) \Omega_3
\]
\[
B_{4}^{D_3} = [B, y_4] + \eta_4(\psi_{11}(0) + \psi_{12}(0) + \psi_{13}(0)) \Omega_4
\]
for some matrices \(y_1, y_2, y_3, y_4 \in \text{Mat}_{4 \times 4}\). Since the matrix \( \Theta \) (defined using (16)) is diagonal, the four rows are needed to span the rowspace of \( \Theta \). Hence,
\[
\text{Mat}_{4 \times 4}^{D_3} = T_{\mathcal{D}_3} \sum_{\mathcal{D}_3} \oplus C \cdot \{ B_{1}^{D_3}, B_{2}^{D_3}, B_{3}^{D_3}, B_{4}^{D_3} \}
\]
and we need to consider all matrices \( \mathcal{R}^j \) \((j = 1, 2, 3, 4)\) in order to compute the \( \mathcal{D}_3 \)-equivariant \( \Lambda \)-mini-versal unfolding.

**Computation of the matrices \( A_j^n \).** We now solve for the matrices \( A_j^n \) using equation (26). Since
\[
A_j^n \Phi(\tau_j) = (a_j^n + 2b_j^n) \Phi(\tau_j),
\]
it is sufficient to solve the first line of (26) to obtain the matrices \( A_j^n \). Consider the first line of equation (26) for \( \mathcal{R}^1 \)
\[
(\eta_1, 0, 0, 0) = \sum_{j=0}^{3} (a_j^1, b_j^1, b_j^1) \Phi(\tau_j).\]
This equation can be rewritten as
\[
(a_0^1 + 2b_0^1, a_1^1 + 2b_1^1, a_2^1 + 2b_2^1, a_3^1 + 2b_3^1) M^t, \tag{29}
\]
where
\[
M = \begin{pmatrix}
e^{i\omega_1 \tau_0} & e^{i\omega_1 \tau_1} & e^{i\omega_1 \tau_2} & e^{i\omega_1 \tau_3} \\
e^{-i\omega_1 \tau_0} & e^{-i\omega_1 \tau_1} & e^{-i\omega_1 \tau_2} & e^{-i\omega_1 \tau_3} \\
e^{i\omega_2 \tau_0} & e^{i\omega_2 \tau_1} & e^{i\omega_2 \tau_2} & e^{i\omega_2 \tau_3} \\
e^{-i\omega_2 \tau_0} & e^{-i\omega_2 \tau_1} & e^{-i\omega_2 \tau_2} & e^{-i\omega_2 \tau_3}
\end{pmatrix}. \tag{30}
\]
 Obviously, \( \det M = 0 \) if any two of the delays are equal or if \( \omega_1 = \omega_2 \). Now, since we take our four delays to be distinct and it is assumed that \( \omega_1 \neq \omega_2 \), then generically \( M \) is nonsingular. Thus, we obtain \( L_1 = \sum_{j=0}^{3} A_j^1 z(\tau_j) \) by solving (29). Now, the equation for \( \mathcal{R}^2 \) yields,
\[
(0, \eta_1, 0, 0) = (a_0^2 + 2b_0^2, a_1^2 + 2b_1^2, a_2^2 + 2b_2^2, a_3^2 + 2b_3^2) M^t,
\]
which if we interchange rows 1 with 2 and 3 with 4 corresponds to the complex conjugate of system (29). So, we can choose \( a_j^2 \) and \( b_j^2 \) so that \( L_2 = \overline{L}_1 \). In a similar manner we obtain \( L_3 \) and \( L_4 = \overline{L}_3 \).

Looking at equation (29), we see that for each \( j = 0, 1, 2, 3, 4 \), either \( a_j^1 \) or \( b_j^1 \) can be chosen arbitrarily. We exploit this freedom to choose constants \( a_j^m \) and \( b_j^m \) which preserve the structure of equation (24).

**\( \mathcal{D}_3 \)-equivariant \( \Lambda \)-unfolding.** Take \( \tau_0 = 0, \tau_1 = \tau_0^*, \tau_2 = \tau_1^*, \tau_3 = \tau_2^*, \tau_4 = \tau_3^*, b_0^m = 0, b_1^m = 0 \) and \( a_2^m = 0 \) for \( m = 1, 2, 3, 4 \) (recall that we have assumed \( \tau_0^*, \tau_1^* \) are nonzero and
distinct so that matrix $M$ in (30) is generically nonsingular with this substitution).

Hence,
\[
L_1 z = a_1^* I z(0) + a_1^* I z(\tau_n^*) + \begin{pmatrix} 0 & b_1^* 1 \ 0 & 0 \end{pmatrix} z(\tau_n^*) + a_1^* I z(\tau_3) + \begin{pmatrix} 0 & b_1^* 1 \ 0 & 0 \end{pmatrix} z(\tau_3)
\]
\[
L_3 z = a_3^* I z(0) + a_3^* I z(\tau_n^*) + \begin{pmatrix} 0 & b_3^* 1 \ 0 & 0 \end{pmatrix} z(\tau_n^*) + a_3^* I z(\tau_3) + \begin{pmatrix} 0 & b_3^* 1 \ 0 & 0 \end{pmatrix} z(\tau_3),
\]

and the complex $\mathbb{D}_3$-equivariant $\Lambda$-versal unfolding is defined by
\[
\mathcal{L}(\alpha) z = \mathcal{L}_0 z + \sum_{m=1}^{4} \alpha_m L_m
\]
\[
= \mathcal{L}_0 z + \sum_{m=1}^{4} \alpha_m a_m^0 I z(0) + \sum_{m=1}^{4} \alpha_m a_m^1 I z(\tau_n^*) + \sum_{m=1}^{4} \alpha_m \begin{pmatrix} 0 & b_m^* 1 \ 0 & 0 \end{pmatrix} z(\tau_n^*)
\]
\[
+ \sum_{m=1}^{4} \alpha_m a_m^2 I z(\tau_3) + \sum_{m=1}^{4} \alpha_m \begin{pmatrix} 0 & b_m^* 1 \ 0 & 0 \end{pmatrix} z(\tau_3).
\]

By the form of equation (29), we can simplify the unfolding even more by setting
for instance $a_m^0 = 0$ for $m = 1, 2, 3, 4$. Thus a $\mathbb{D}_3$-equivariant $\Lambda$-versal unfolding
of (24) near $(\alpha^*, \beta^*, \tau_n^*, \tau_n^*)$ is given by
\[
\dot{u}_1(t) = (-1 + \epsilon_1) u_1(t) + (\alpha^* + \epsilon_2) u_1(t - \tau_n^*) + (\beta^* + \epsilon_3) (u_3(t - \tau_n^*) + u_2(t - \tau_n^*))
\]
\[
+ \epsilon_4 (u_3(t - \tau_3) + u_2(t - \tau_3))
\]
\[
\dot{u}_2(t) = (-1 + \epsilon_1) u_2(t) + (\alpha^* + \epsilon_2) u_2(t - \tau_n^*) + (\beta^* + \epsilon_3) (u_1(t - \tau_n^*) + u_3(t - \tau_n^*))
\]
\[
+ \epsilon_4 (u_1(t - \tau_3) + u_3(t - \tau_3))
\]
\[
\dot{u}_3(t) = (-1 + \epsilon_1) u_3(t) + (\alpha^* + \epsilon_2) u_3(t - \tau_n^*) + (\beta^* + \epsilon_3) (u_2(t - \tau_n^*) + u_1(t - \tau_n^*))
\]
\[
+ \epsilon_4 (u_2(t - \tau_3) + u_1(t - \tau_3)).
\]

where $\epsilon_1 = \sum_{m=1}^{4} \alpha_m a_m^0$, $\epsilon_2 = \sum_{m=1}^{4} \alpha_m a_m^1$, $\epsilon_3 = \sum_{m=1}^{4} \alpha_m b_m^0$ and $\epsilon_4 = \sum_{m=1}^{4} \alpha_m b_m^1$.

Now, the parameters $\epsilon_i$ are linearly independent since
\[
\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix} = \begin{pmatrix} a_1^0 & a_1^0 & a_1^0 & a_1^0 \\ a_1^0 & a_1^0 & a_1^0 & a_1^0 \\ b_2^0 & b_2^0 & b_2^0 & b_2^0 \\ b_3^0 & b_3^0 & b_3^0 & b_3^0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = M^{-1} \text{diag}(\eta_1, \eta_1, \eta_1, \eta_1) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}.
\]

We now compute the real $\mathbb{D}_3$-equivariant $\Lambda$-versal unfolding. From [3] we know that the real unfolding is given by
\[
\mathcal{L}(\alpha) = \mathcal{L}_0 + \alpha_1 \text{Re}(L_1) + \alpha_2 \text{Im}(L_1) + \alpha_3 \text{Re}(L_3) + \alpha_4 \text{Im}(L_3)
\]
so the delay-differential equation is identical to (31) with
\[
\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix} = \begin{pmatrix} \text{Re}(a_1^0) \text{Im}(a_0^1) \text{Re}(a_0^1) \text{Im}(a_0^1) \\ \text{Re}(a_1^0) \text{Im}(a_1^0) \text{Re}(a_1^0) \text{Im}(a_1^0) \\ \text{Re}(b_2^0) \text{Im}(b_2^0) \text{Re}(b_2^0) \text{Im}(b_2^0) \\ \text{Re}(b_3^0) \text{Im}(b_3^0) \text{Re}(b_3^0) \text{Im}(b_3^0) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}.
\]
Moreover, the new parameters $\epsilon_i$, $i = 1, 2, 3, 4$ are also linearly independent since the matrix of coefficients of the right-hand side of (33) is nonsingular because the four columns of this matrix span the four dimensional real subspace of the four dimensional complex space spanned by the columns of the matrix in (32).

5.2. Double imaginary eigenvalues. We now consider the case where the eigenvalues of $A$ are double. The space $P$ has basis

$$\Phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta), \phi_5(\theta), \phi_6(\theta), \phi_7(\theta), \phi_8(\theta))$$

where $v = (1, \omega, \omega^2)^t$ and $\omega = e^{2\pi i/3}$. We now compute the basis $\Psi$ of the adjoint $P^*$. The action of $D_3$ on $\mathbb{C}^8$ is generated by matrices

$$G(\gamma) = \text{diag}(e^{2\pi i/3}, e^{-2\pi i/3}, e^{2\pi i/3}, e^{-2\pi i/3}, e^{2\pi i/3}, e^{-2\pi i/3}, e^{2\pi i/3}, e^{-2\pi i/3}), \quad G(\kappa) = \text{diag}(J, J, J)$$

where $J = [0 \ 1 \ 0]$. From equation (10), we know that $\Psi$ commutes with the actions (23) and (34). Now, recall that $\Psi$ is chosen such that $(\Psi, \Phi) = I$ and note that $\phi_3 = \phi_3$, $\phi_4 = \phi_2$, $\phi_7 = \phi_5$, $\phi_8 = \phi_6$, then along with the equivariance condition on $\Psi$ we obtain

$$\Psi(0) = \begin{pmatrix} \mu(1 \ \sigma \ \omega) \\ \mu(1 \ \omega \ \varphi) \\ \mu(1 \ \varphi \ \psi) \\ \mu(1 \ \sigma \ \psi) \\ \mu(1 \ \varphi \ \varphi) \\ \mu(1 \ \sigma \ \sigma) \\ \zeta(1 \ \sigma \ \varphi) \\ \zeta(1 \ \omega \ \varphi) \\ \zeta(1 \ \varphi \ \varphi) \\ \zeta(1 \ \sigma \ \sigma) \end{pmatrix},$$

where $\mu$ and $\zeta$ are some nonzero complex numbers.

Basis of the unfolding space. Using (25), the dimension of the unfolding space $W$ for matrix $B$ is 16 since here $r = 4$ is the number of distinct eigenvalues, $k_j = 2$ is the number of Jordan blocks for each eigenvalue and $n_{j, \ell} = 1$ is the size of each Jordan block.

We now define the 16 matrices which form the basis of $E(W)$. We define $3 \times 8$ matrices $R_{j, \xi, \theta, m}$ for each eigenvalue $\lambda_j$, where $\lambda_1 = \omega_1 i$, $\lambda_2 = -\omega_2 i$, $\lambda_3 = \omega_2 i$, $\lambda_4 = -\omega_1 i$. Let $v_i = (v_{i1}, v_{i2}, v_{i3})^t$, $i = 1, \ldots, 8$ be $3 \times 1$ complex vectors. The linear mappings $\Pi_j$ are

$$\Pi_1(v) = \begin{pmatrix} \mu(v^1 + \omega v^2 + \omega v^3) \\ \mu(v^1 + \omega v^2 + \omega v^1) \end{pmatrix}, \quad \Pi_2(v) = \begin{pmatrix} \nu(v^1 + \omega v^2 + \omega v^3) \\ \nu(v^1 + \omega v^2 + \omega v^1) \end{pmatrix}$$

$$\Pi_3(v) = \begin{pmatrix} \zeta(v^1 + \omega v^2 + \omega v^3) \\ \zeta(v^1 + \omega v^2 + \omega v^1) \end{pmatrix}, \quad \Pi_4(v) = \begin{pmatrix} \zeta(v^1 + \omega v^2 + \omega v^3) \\ \zeta(v^1 + \omega v^2 + \omega v^1) \end{pmatrix}$$

We choose $\xi, \theta \in \{1, 2\}$ and $m = 1$ since each Jordan block is one-dimensional. Then, $R_{j, \xi, \theta, m}$ has vector $v_{2(j-1)+\xi}$ in column $2(j-1)+\theta$ and all the other columns are zero where

$$\Pi_j(v_{2j-1}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Pi_j(v_{2j}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and $v_3 = \overline{v_1}$, $v_4 = \overline{v_2}$, $v_7 = \overline{v_5}$, $v_8 = \overline{v_6}$. 
To simplify the notation, we order the 16 matrices \( R_{ij; \vartheta, m} \) in the following way:
\[
\{ R^{1+8(k-1)}, \ldots, R^{8+8(k-1)} \} = \{ R_{1;1, k, 1}, R_{2;1, k, 1}, R_{3;1, k, 1}, R_{1;2, k, 1}, R_{2;2, k, 1}, R_{3;2, k, 1}, R_{4;2, k, 1} \}
\]
for \( k = 1, 2 \). One can verify using a computer algebra package that
\[
\text{rank}(\Phi(\tau_0), \Phi(\tau_1), \Phi(\tau_2), \Phi(\tau_3)) = 8,
\]
therefore we need four distinct delays \( \tau_0, \tau_1, \tau_2, \tau_3 \) to solve
\[
R^n = \sum_{i=0}^{3} A_i^n \Phi(\tau_i).
\] (37)

**Projection of the basis.** We project the matrices \( R^n \) on the subspace of \( \mathbb{D}_3 \)-equivariant \( 3 \times 8 \) matrices, using the actions on \( \mathbb{C}^3 \) (23) and \( \mathbb{C}^8 \) (34).

**Proposition 5.1.** Let \( \nu = (\nu^1, \nu^2, \nu^3)^t \in \mathbb{C}^3 \) and consider the \( 3 \times 8 \) matrix \( N_p \) having \( \nu \) as its \( p^{th} \) column and all the other columns are zero. From (27), we obtain a projection operator from which we compute
\[
\mathcal{N}_p = \sum_{g \in \mathbb{D}_3} \rho(g)N_pG(g^{-1}).
\]
Moreover, let \( \eta_1(\nu) = \nu^1 + \nu^2 + \nu^3 \), \( \eta_2(\nu) = \nu^1 + \nu^2 + \nu^3 \) and recall that \( v = (1, \omega, \overline{\omega})^t \).

1. If \( p = 1, 5 \), then \( \mathcal{N}_p \) has only nonzero columns \( p \) and \( p+1 \) given respectively by \( \eta_1 v \) and \( \eta_1 \overline{v} \),
2. if \( p = 4, 8 \), then \( \mathcal{N}_p \) has only nonzero columns \( p-1 \) and \( p \) given respectively by \( \eta_1 \overline{v} \) and \( \eta_1 v \),
3. if \( p = 3, 7 \), then \( \mathcal{N}_p \) has only nonzero columns \( p \) and \( p+1 \) given respectively by \( \eta_2 v \) and \( \eta_2 \overline{v} \),
4. and if \( p = 2, 6 \), then \( \mathcal{N}_p \) has only nonzero columns \( p-1 \) and \( p \) given respectively by \( \eta_2 \overline{v} \) and \( \eta_2 v \).

**Proof.** The proof is a straightforward calculation. \( \square \)

From the definition of the matrices \( R^n \), for \( m = 1, \ldots, 8 \) we have \( \vartheta = 1 \) and for \( m = 9, \ldots, 16 \) we have \( \vartheta = 2 \). Table 1 shows the correspondence of matrices \( R^n \) with matrices \( N_p \) of Proposition 5.1 as well as the information given by (35) and (36) needed to compute \( \mathcal{R}^m \). From Proposition 5.1 and Table 1, we obtain easily that \( \mathcal{R}^m = 0 \) for \( m = 5, \ldots, 12 \) and
\[
\begin{align*}
\mathcal{R}^1 &= \mathcal{R}^{13} = [\mu^{-1} v, \mu^{-1} \overline{v}, 0, 0, 0, 0, 0, 0], & \mathcal{R}^2 &= \mathcal{R}^{14} = [0, 0, \mu^{-1} v, \mu^{-1} \overline{v}, 0, 0, 0, 0], \\
\mathcal{R}^3 &= \mathcal{R}^{15} = [0, 0, 0, 0, \zeta^{-1} v, \zeta^{-1} \overline{v}, 0, 0], & \mathcal{R}^4 &= \mathcal{R}^{16} = [0, 0, 0, 0, 0, 0, \zeta^{-1} v, \zeta^{-1} \overline{v}].
\end{align*}
\]

**\( \mathbb{D}_3 \)-mini-versal unfolding** Since \( \mathcal{E}(\mathcal{W}) \) is an unfolding space for \( B \), we again set \( \mathcal{B}_j^{\mathbb{D}_3} = \Psi(0)\mathcal{R}^j \) for \( j = 1, \ldots, 16 \). The space \( \mathcal{W} \) is spanned by sixteen \( 8 \times 8 \) matrices. For \( i = 1, \ldots, 8 \), \( \Omega_i \) has a 1 at element \((i, i)\) and zeroes elsewhere. We now list the remaining matrices along with the location of its element 1: \( \Omega_0; (1, 2), \Omega_{10}; (2, 1), \Omega_{11}; (3, 4), \Omega_{12}; (4, 3), \Omega_{13}; (5, 6), \Omega_{14}; (6, 5), \Omega_{15}; (8, 7), \Omega_{16}; (7, 8) \). Now, obviously,
\[ B_j^{\mathbb{D}_3} = 0 \] for \( j = 5, \ldots, 12 \) and a computation shows that\[ B_1^{\mathbb{D}_3} = B_{13}^{\mathbb{D}_3} = [B, y_1] + (1 + (1 + \mu^{-1})|\omega|^2)\Omega_1 + (1 + 2|\omega|^2)\Omega_2 + (1 + \omega^2 + \mu^{-1}\omega^2)\Omega_9 + (1 + \omega^2 + \overline{\omega}^2)\Omega_{10} \]
\[ B_2^{\mathbb{D}_3} = B_{14}^{\mathbb{D}_3} = [B, y_2] + (1 + 2|\omega|^2)\Omega_3 + (1 + 2|\omega|^2)\Omega_4 + (1 + \omega^2 + \overline{\omega}^2)(\Omega_{11} + \Omega_{12}) \]
\[ B_3^{\mathbb{D}_3} = B_{15}^{\mathbb{D}_3} = [B, y_3] + (1 + (1 + \zeta^{-1})|\omega|^2)\Omega_5 + (1 + 2|\omega|^2)\Omega_6 + (1 + \omega^2 + \zeta^{-1}\omega^2)\Omega_{13} + (1 + \omega^2 + \overline{\omega}^2)\Omega_{14} \]
\[ B_4^{\mathbb{D}_3} = B_{16}^{\mathbb{D}_3} = [B, y_4] + (1 + 2|\omega|^2)\Omega_7 + (1 + 2|\omega|^2)\Omega_8 + (1 + \omega^2 + \overline{\omega}^2)(\Omega_{15} + \Omega_{16}). \]

for some matrices \( y_1, y_2, y_3, y_4 \in \text{Mat}_{8 \times 8}. \) The four first rows of \( \Theta \) are:
\[ v_1 = (1 + (1 + \mu^{-1})|\omega|^2, 1 + 2|\omega|^2, 0, 0, 0, 0, 0, 1 + \omega^2 + \mu^{-1}\omega^2, 1 + \omega^2 + \overline{\omega}^2, 0, 0, 0, 0, 0, 0, 0) \]
\[ v_2 = (0, 0, 1 + 2|\omega|^2, 0, 0, 0, 0, 0, 1 + \omega^2 + \overline{\omega}^2, 1 + \omega^2 + \overline{\omega}^2, 0, 0, 0, 0) \]
\[ v_3 = (0, 0, 0, 0, 1 + (1 + \zeta^{-1})|\omega|^2, 1 + 2|\omega|^2, 0, 0, 0, 0, 1 + \omega^2 + \zeta^{-1}\omega^2, 1 + \omega^2 + \overline{\omega}^2, 0, 0) \]
\[ v_4 = (0, 0, 0, 0, 0, 1 + 2|\omega|^2, 1 + 2|\omega|^2, 0, 0, 0, 0, 1 + \omega^2 + \overline{\omega}^2, 1 + \omega^2 + \overline{\omega}^2), \]

rows 5 to 12 are zero and rows 13 to 16 are identical to rows 1 to 4. Thus, \( \{v_1, v_2, v_3, v_4\} \) determine a maximal set of linearly independent vectors in the rowspace of \( \Theta \). Hence,

\[ \text{Mat}_{8 \times 8} = T_B \Sigma^{\mathbb{D}_3} \oplus \mathbb{C} \cdot \{B_1^{\mathbb{D}_3}, B_2^{\mathbb{D}_3}, B_3^{\mathbb{D}_3}, B_4^{\mathbb{D}_3}\} \]

and we need only consider matrices \( R_j \) for \( j = 1, 2, 3, 4 \) in order to compute the \( \mathbb{D}_3 \)-equivariant \( \Lambda \)-mini-versal unfolding.

**Computation of the matrices \( A_j^m \).** We take \( \mathbb{D}_3 \)-equivariant matrices \( A_j^m \) as in (28) and we compute

\[ A_j^m \Phi(\tau_j) = (a_j^m + b_j^m(\overline{\omega} + \omega)) \Phi(\tau_j). \]
Thus, equations $R^m = \sum_{j=0}^3 A^m_j \Phi(\tau_j)$ for $m = 1, \ldots, 4$ can be solved by finding the solution to the first row only

\[
(\mu^{-1}, 0, 0, 0) = (a_1^1 + b_0^1(\omega + \omega), a_1^1 + b_1^1(\omega + \omega), a_2^1 + b_2^1(\omega + \omega), a_3^1 + b_3^1(\omega + \omega)) M^t \quad (38)
\]

where $M$ is the generically nonsingular matrix (30).

**Remark 5.2.** Note that, as in the previous example, from the form of equations (38) we can choose $a_j^2 = \tau_j^1$, $a_j^3 = \tau_j^2$, $b_j^2 = \bar{b}_j$, and $b_j^4 = \bar{b}_j$ for $j = 0, 1, 2, 3$.

**$\mathbb{D}_3$-equivariant $\Lambda$-unfolding.** For $m = 1, 2, 3, 4$, define

\[
L_m z = \sum_{j=0}^3 A^m_j z(\tau_j) = \sum_{j=0}^3 \left( \begin{array}{ccc} a_j^m & b_j^m & b_j^m \\ b_j^m & a_j^m & b_j^m \\ b_j^m & b_j^m & a_j^m \end{array} \right) z(\tau_j),
\]

and then the complex $\mathbb{D}_3$-equivariant $\Lambda$-versal unfolding is defined by:

\[
L(\alpha) z = L_0 z + \sum_{j=0}^3 \left( \begin{array}{ccc} 0 & \epsilon_1 & \epsilon_2 \\ \epsilon_2 & \epsilon_3 & \epsilon_4 \\ \epsilon_4 & 0 & \epsilon_3 \end{array} \right)
\]

As in the previous example, setting $\tau_0 = 0$, $\tau_1 = \tau_2^\ast$, $\tau_2 = \tau_3^\ast$ and $b_0^m = b_1^m = a_2^m = a_3^m = 0$ for $m = 1, 2, 3, 4$ preserves the structure of the delay-differential equation. Then as a delay-differential equation, the complex unfolding is given exactly by (31) with again $\epsilon_1 = \sum_{m=1}^4 \alpha_m a_0^m$, $\epsilon_2 = \sum_{m=1}^4 \alpha_m a_1^m$, $\epsilon_3 = \sum_{m=1}^4 \alpha_m b_2^m$, and $\epsilon_4 = \sum_{m=1}^4 \alpha_m b_3^m$.

Note that the unfolding parameters $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ are linearly independent by Remark 5.2 and from the same calculation as in the previous example. Moreover, from Remark 5.2, the real $\mathbb{D}_3$-equivariant $\Lambda$-versal unfolding is also given by (31).

**Appendix A. Proof of Lemma 3.3.** We will prove the result in the case where (13) holds with $p$ equal to the codimension of $T_B \Sigma^\Gamma$ in $\text{Mat}_{c\times c}^\Gamma$. The proof follows along the same lines as the non-equivariant case (which can be found, for example in [1]). We first define the $\Gamma$-centralizer of $u \in \text{Mat}_{c\times c}^\Gamma$ as the linear subspace $Z_u^\Gamma$ of all matrices in $\text{Mat}_{c\times c}^\Gamma$ which commute with $u$. Consider now the $\Gamma$-centralizer $Z_B^\Gamma$ of the matrix $B$. In the space $GL(c, \mathbb{C}) \cap \text{Mat}_{c\times c}^\Gamma$, let $P$ be a smooth surface transversal to $I_{c\times c} + Z_B^\Gamma$ at $I_{c\times c}$, and such that the dimension of $P$ is equal to the codimension of $Z_B^\Gamma$ in $\text{Mat}_{c\times c}^\Gamma$. We then define a mapping

\[
F : P \times \mathbb{C}^p \longrightarrow \text{Mat}_{c\times c}^\Gamma, \quad F(p, \alpha) = p B(\alpha) p^{-1}.
\]

The key result we need is

**Lemma A.1.** $F$ is a local diffeomorphism in a neighborhood of $(I_{c\times c}, \alpha_0)$.

**Proof.** First, we note that the mapping

\[
f : GL(c, \mathbb{C}) \cap \text{Mat}_{c\times c}^\Gamma \longrightarrow \Sigma^\Gamma \subset \text{Mat}_{c\times c}^\Gamma,
\]

\[
f(C) = CBC^{-1}
\]
is such that the derivative of $f$ evaluated at $I_{\alpha\chi}$ is

$$Df(I_{\alpha\chi}) : T_{I_{\alpha\chi}} \text{Mat}_{\alpha\chi} \cong \text{Mat}_{\alpha\chi}^\Gamma \to T_B \Sigma_p \subset T_B \text{Mat}_{\alpha\chi} \cong \text{Mat}_{\alpha\chi}^\Gamma,$$

which is onto $T_B \Sigma_p$. It follows from the above that the dimension of $Z_p^\Gamma$ is equal to the codimension of $T_B \Sigma_p$ in $\text{Mat}_{\alpha\chi}^\Gamma$, which is equal to $p$. Also, the dimension of $\mathcal{P}$ is equal to the dimension of $T_B \Sigma_p$.

Now, the mapping $\mathcal{F}$ is such that

$$D_B \mathcal{F}(I_{\alpha\chi}, \alpha_0)(u, \alpha) = Df(I_{\alpha\chi})(u) = [u, B], \quad D_B \mathcal{F}(I_{\alpha\chi}, \alpha_0)(u, \alpha) = D_B \mathcal{B}(\alpha_0) \alpha.$$

By construction of $\mathcal{P}$, we have that

$$D_B \mathcal{F}(I_{\alpha\chi}, \alpha_0) = Df(I_{\alpha\chi})|_{T_{I_{\alpha\chi}} \mathcal{P}}$$

maps $T_{I_{\alpha\chi}} \mathcal{P}$ isomorphically onto $T_B \Sigma_p$. Also, by the hypothesis of Lemma 3.3, $D_B \mathcal{B}(\alpha_0)$ maps $T_{\alpha_0} \mathcal{C}^p \cong \mathcal{C}^p$ isomorphically onto a space which is a direct sum complement of $T_B \Sigma_p$ in $\text{Mat}_{\alpha\chi}^\Gamma$. It thus follows that $D_B \mathcal{F}(I_{\alpha\chi}, \alpha_0)$ is an isomorphism between vector spaces of dimension $\dim(\text{Mat}_{\alpha\chi}^\Gamma)$, so $\mathcal{F}$ is a local diffeomorphism by the inverse function theorem. \hfill \Box

Now, let $A(\beta)$ be a $q$-parameter $\Gamma$-unfolding of $B$ (with $A(\beta_0) = B$). Define $\Pi_1$ and $\Pi_2$ as the projections of $\mathcal{P} \times \mathcal{C}^p$ onto $\mathcal{P}$ and $\mathcal{C}^p$ respectively.

For all $\beta$ sufficiently close to $\beta_0$ in $\mathcal{C}^q$, define

$$C(\beta) = \Pi_1 \mathcal{F}^{-1}(A(\beta)) \quad \text{and} \quad \phi(\beta) = \Pi_2 \mathcal{F}^{-1}(A(\beta)).$$

It follows that for all $\beta$ sufficiently close to $\beta_0$,

$$A(\beta) = \mathcal{F}(C(\beta), \phi(\beta)) = C(\beta) \mathcal{B}(\phi(\beta))(C(\beta))^{-1},$$

which proves the lemma.

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