Realization of Critical Eigenvalues for Scalar and Symmetric Linear Delay-Differential Equations

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Abstract. This paper studies the link between the number of critical eigenvalues and the number of delays in certain classes of delay-differential equations. There are two main results. The first states that for \( k \) purely imaginary numbers which are linearly independent over the rationals, there exists a scalar delay-differential equation depending on \( k \) fixed delays whose spectrum contains those \( k \) purely imaginary numbers. The second result is a generalization of the first result for delay-differential equations which admit a characteristic equation consisting of a product of \( s \) factors of scalar type. In the second result, the \( k \) eigenvalues can be distributed among the different factors. Since the characteristic equation of scalar equations contain only exponential terms, the proof exploits a toroidal structure which comes from the arguments of the exponential terms in the characteristic equation. Our second result is applied to delay coupled \( D_n \)-symmetric cell systems with one-dimensional cells. In particular, we provide a general characterization of delay coupled \( D_n \)-symmetric systems with an arbitrary number of delays and cell dimension.

Key words. bifurcation, delay-differential equations, symmetry, realizability

AMS subject classifications. 37G10, 37G40, 34K06, 34K18

DOI. 10.1137/08071363X

1. Introduction and background. Delay-differential equations (DDEs) have been used as mathematical models for phenomena in population dynamics [21], physiology [12, 3], physics [23], climate modeling [26, 28], and engineering [27], among others. DDEs behave like abstract ordinary differential equations (ODEs) on an infinite-dimensional (Banach) phase space and many results which are known for ODEs on finite-dimensional spaces have analogues in the context of DDEs. Many scalar DDE models have been developed over the years, such as for Cheynes–Stokes respiration [12] and the regulation of hematopoiesis [12], the delayed Nicholson blowflies equation [18] in population dynamics, a two-delay model of an experiment on Parkinsonian tremor [2], and many more.

The bifurcation analysis of DDEs is done essentially in the same way as that of ODEs, although the technical details differ. Consider the neighborhood of an equilibrium solution of a nonlinear DDE; then the analysis of the linearization at the equilibrium point leads to stable, unstable, and center invariant subspaces where only the stable subspace is infinite-dimensional. There exist local invariant manifolds (stable, unstable, and center manifolds) tangent to the
corresponding invariant subspaces of the linearized equations about the equilibrium point on which the flow near the equilibrium is exponentially attracting (stable manifold), exponentially repelling (unstable manifold), or nonhyperbolic (center manifold). Now, bifurcations near equilibria are determined by the flow on the center manifold, and the dimension of this manifold is determined by the number of eigenvalues of the linearization on the imaginary axis.

The first result of our paper is Theorem 2.1 and goes as follows. Consider \( n \) nonzero imaginary numbers \( i\omega_1, \ldots, i\omega_n \), where the imaginary parts \( \omega_1, \ldots, \omega_n \) are positive and not rationally dependent. We show that there exists a scalar linear DDE depending on \( n \) discrete delays written

\[
\dot{x} = \sum_{j=1}^{n} a_j x(t - \tau_j),
\]

where \( x \in \mathbb{R}, a_j \in \mathbb{R}, \) and \( \tau_j \in [0, \tau] \) for all \( j = 1, \ldots, n \) such that the characteristic equation of (1), given by

\[
\lambda - \sum_{j=1}^{n} a_j e^{-\lambda \tau_j} = 0,
\]

has eigenvalues \( \pm i\omega_1, \ldots, \pm i\omega_n \). This result generalizes explicit computations done in the case of one and two delays; see [20, 9, 1]. The proof is done by embedding the problem as a mapping which is solved by the implicit function theorem at a carefully chosen point. From the implicit function theorem, we are able to define a smooth mapping whose transversal intersection with a dense curve on an \( n \)-dimensional torus provides solutions. The incommensurability of the \( n \) frequencies enables us to define the dense curve on the \( n \)-torus. This type of argument using a dense curve on an \( n \)-dimensional torus was used in Choi and LeBlanc [7].

This result falls within the category of so-called realization theorems, for instance, the realization theorem of linear ODEs by linear DDEs obtained by Faria and Magalhães [11]. They show that for any finite-dimensional matrix \( B \), a necessary and sufficient condition for the existence of a bounded linear operator \( L_0 \) from \( C([-\tau, 0], \mathbb{R}^n) \) into \( \mathbb{R}^n \) with infinitesimal generator having spectrum containing the spectrum of \( B \) is that \( n \) be larger than or equal to the largest number of Jordan blocks associated with each eigenvalue of \( B \). Other results in this direction are concerned with the realization of finite jets of ODEs on a finite-dimensional center manifold by DDEs; see [11, 7]. To our knowledge, the realization theorems in this paper are the first general results linking the number of critical eigenvalues of linear DDEs with the number of discrete delays.

The next significant result is an openness theorem, that is, the realization of \( n \) imaginary numbers (not necessarily rationally independent) as eigenvalues of a linear scalar DDE is valid in a neighborhood of any set of \( n \) rationally independent imaginary numbers. The proof of this theorem also relies on the implicit function theorem.

We then turn our attention to the context of symmetric systems of DDEs. Several examples of symmetric systems of DDEs [16, 24] have characteristic equations which decompose in factors, some of which have the same form as the characteristic equation (2). The decomposition of the characteristic equation is induced by the isotypic decomposition of the space,
and we present a general derivation of this decomposition. We show that isotypic components consisting of a unique one-dimensional complex irreducible representation contribute a factor of the form (2) in the characteristic equation, and so Theorem 2.1 can be applied directly to each such factor separately.

We present a generalization of Theorem 2.1 to the case where several factors of the characteristic equation have purely imaginary eigenvalues simultaneously. Theorem 2.4 shows that a set of \( n \) rationally independent purely imaginary complex numbers can be realized from several factors of the characteristic equation of a DDE with \( n \) delays given that two nondegeneracy conditions on the characteristic equation are satisfied. The statement of the theorem is independent of any symmetric structure, and the proof is a generalization of the proof of Theorem 2.1.

We illustrate the above result on \( D_n \)-symmetric rings of \( n \) delay equations with delayed coupling. Hopf bifurcation from such symmetric networks has been studied by several authors [6, 10, 15, 16, 17, 24, 25, 29, 30]. We study only the case in which \( n \) is odd, since, for \( n \) even, one of the nondegeneracy conditions of Theorem 2.4 is not always satisfied, as we illustrate in a \( D_4 \) example.

In order to apply Theorem 2.4 to this context, we derive an explicit form of the coupling matrix in terms of the connections in the graph representation of the ring for cells of any dimension and arbitrary numbers of connections and delays. This is a generalization of the networks considered in the articles listed in the previous paragraph. We specialize to the case of one-dimensional cells and show how Theorem 2.4 applies to \( D_n \)-symmetric coupled cell systems with \( n \) odd.

The paper is organized as follows. The first section contains brief preliminary remarks, and then we state and prove our main result (Theorem 2.1) and the openness result (Theorem 2.2). Then we introduce the context leading to Theorem 2.4 and state this result. Section 3 is devoted to \( \Gamma \)-symmetric systems of DDEs, and the section begins with a general discussion. Section 3.1 presents a characterization of \( D_n \)-symmetric rings of delay coupled cells with an arbitrary number of delays, and the characteristic equation in the case of one-dimensional cells is derived. Theorem 2.4 is applied to \( D_n \)-symmetric rings of one-dimensional cells with \( n \) odd. Section 4 contains the proof of Theorem 2.4. We conclude with a discussion of open problems along the lines of those presented in this paper.

2. Realization theorems. We now discuss some aspects of the spectral theory of linear scalar DDEs. In fact, we just introduce the basic facts, in a nonabstract setting, needed for the statement of our first main theorem. For a complete treatment, see Diekmann et al. [9] or Hale and Verduyn-Lunel [20].

Consider the scalar DDE

\[
\dot{x}(t) = \sum_{j=1}^{n} a_j x(t - \tau_j),
\]

where \( a_j \in \mathbb{R} \) and \( \tau_j \in [0, \tau] \) for all \( j = 1, \ldots, n \) and \( \tau > 0 \). The characteristic equation for (3) can be obtained by substituting \( x(t) = Ce^{\lambda t} \), where \( C \) is a constant, into the equation. Thus,

\[
\lambda Ce^{\lambda t} = \sum_{j=1}^{n} a_j Ce^{\lambda(t-\tau_j)} = \sum_{j=1}^{n} a_j Ce^{-\lambda \tau_j} e^{\lambda t},
\]
and by rearranging the terms we obtain
\[ \left( \lambda - \sum_{j=1}^{n} a_j e^{-\lambda \tau_j} \right) x(t) = 0. \]

So, \( x(t) \) is a nonzero solution of (3) if and only if
\[ \Delta(\lambda) := \lambda - \sum_{j=1}^{n} a_j e^{-\lambda \tau_j} = 0. \]

The complex number \( \lambda \) is an eigenvalue of equation (3) if it is a solution of the characteristic equation \( \Delta(\lambda) = 0 \).

The question we address in this paper is related to the number of imaginary eigenvalues (with incommensurable frequencies) which can satisfy \( \Delta(\lambda) = 0 \). The case \( n = 1 \) with one nonzero delay is a straightforward calculation and \( \Delta(\lambda) = 0 \) for only one nonzero imaginary eigenvalue; see [20]. The case \( n = 2 \) with \( \tau_1 = 0 \) and \( \tau_2 \in (0, \tau] \) in (3) can be found in [9]. There, it is shown that \( \Delta(\lambda) = 0 \) can have at most two nonzero imaginary eigenvalues. The case \( n = 2 \) with \( \tau_1, \tau_2 > 0 \) is done in [1], where it is shown that \( \Delta(\lambda) = 0 \) can have at most two nonzero imaginary eigenvalues. We are now ready to state our first result.

**Theorem 2.1.** Suppose \( \omega_1 > 0, \omega_2 > 0, \ldots, \omega_n > 0 \) are linearly independent over the rationals. Then there exist \( \tau_1 > 0, \tau_2 > 0, \ldots, \tau_n > 0, a_1 \in \mathbb{R}, a_2 \in \mathbb{R}, \ldots, a_n \in \mathbb{R} \) such that the linear DDE
\[ \dot{x}(t) = a_1 x(t - \tau_1) + a_2 x(t - \tau_2) + \cdots + a_n x(t - \tau_n) \]
has solutions \( x^\pm_j(t) = e^{\pm i\omega_j t} \) for all \( j = 1, \ldots, n \).

**Proof.** A necessary and sufficient condition for the conclusion of the theorem to hold is that the algebraic system of \( 2n \) equations
\[ \sum_{k=1}^{n} a_k e^{-i\omega_j \tau_k} = i\omega_j, \quad j = 1, \ldots, n, \]
\[ \sum_{k=1}^{n} a_k e^{i\omega_j \tau_k} = -i\omega_j, \quad j = 1, \ldots, n, \]
has a solution in the \( 2n \) unknowns \( (\tau_1, \tau_2, \ldots, \tau_n, a_1, a_2, \ldots, a_n) \). Although (5) is in complex form, since the second equation in (5) is just the complex conjugate of the first equation in (5), system (5) is equivalent to a system of \( 2n \) real equations. This fact is taken for granted throughout what follows, even though we continue to use complex notation.

It is useful to use the following matrix notation for (5):
\[ \begin{pmatrix} P(\tau; \omega) \\ P(-\tau; \omega) \end{pmatrix} A^T = \begin{pmatrix} i\omega^T \\ -i\omega^T \end{pmatrix}, \]
where \( \tau = (\tau_1, \ldots, \tau_n), \omega = (\omega_1, \ldots, \omega_n), A = (a_1, \ldots, a_n) \), superscript \( T \) denotes transpose, and \( P(\tau; \omega) = P(\tau_1, \ldots, \tau_n; \omega_1, \ldots, \omega_n) \) is the \( n \times n \) matrix whose entry at row \( j \) column \( k \) is
\[ [P(\tau; \omega)]_{jk} = e^{-i\omega_j \tau_k}. \]
Note that \( P(\tau;\omega) = P(-\tau;\omega) \).

Instead of attempting to solve (6) directly, we adopt an approach based on the following fact. For \( j, k = 1, \ldots, n \), consider the exponents \( \omega_j \tau_k \) in \( P(\tau;\omega) \) taken modulo \( 2\pi \). Since the \( \omega_j \) are rationally independent, for \( \tau_k \geq 0 \), the vector \( \tau_k \omega \mod 2\pi \) generates a dense orbit, denoted by \( O_k \), on an \( n \)-torus \( \mathbb{T}^n \), where \( \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \). If \( (\tau, A) \) is a solution to (6), then \( \tau \) produces a point on \( V := (\mathbb{T}^n)^n \) via the dense orbits.

Thus, we embed the problem of finding solutions of (6) into the problem of finding solutions of a mapping \( F \) defined on \( V \) and which is an extension of (6). The idea is that an explicit solution of \( F = 0 \) is easily obtained, and we use the implicit function theorem to find a submanifold of solutions to \( F = 0 \). We then show that the structure of the dense orbits \( O_k \) on this submanifold yields an infinite number of solutions to \( F = 0 \) and therefore to (6).

Choose coordinates on \( V \) as follows:

\[
V := \{ \Phi = (\Phi^1, \ldots, \Phi^n) \mid \Phi^j = (\varphi^j_1, \ldots, \varphi^j_n) \in \mathbb{T}^n, j = 1, \ldots, n \},
\]

and consider the following mapping associated to (6):

\[
F : V \times \mathbb{R}^n \rightarrow \mathbb{R}^{2n}
\]

defined by

\[
(7) \hspace{1cm} F(\Phi, A; \omega) = \left( \begin{array}{c} \tilde{P}(\Phi) \\ \tilde{P}(-\Phi) \end{array} \right) A^T - i \left( \begin{array}{c} \omega^T \\ -\omega^T \end{array} \right),
\]

where \( A \) and \( \omega \) are as previously defined and \( \tilde{P}(\Phi) \) is the \( n \times n \) matrix whose entry at row \( j \) column \( k \) is

\[
\left[ \tilde{P}(\Phi) \right]_{jk} = \left[ \tilde{P}(\Phi^1, \ldots, \Phi^n) \right]_{jk} = e^{-i\varphi^j_k}.
\]

Letting \( \Psi = \Phi^n \) is a convenient notation to use when applying the implicit function theorem, i.e.,

\[
\Psi = (\psi_1, \ldots, \psi_n) = \Phi^n = (\varphi^n_1, \ldots, \varphi^n_n).
\]

We write \( V = V_\Phi \times V_\Psi \), where \( V_\Phi \cong (\mathbb{T}^n)^{n-1} \) and \( V_\Psi \cong \mathbb{T}^n \) so that

\[
F : V_\Phi \times V_\Psi \times \mathbb{R}^n \rightarrow \mathbb{R}^{2n}
\]
is written as \( F(\Phi, \Psi, A; \omega) \) in (7) (we have relabeled \( \Phi = (\Phi^1, \ldots, \Phi^{n-1}) \) to designate coordinates for \( V_\Phi \cong (\mathbb{T}^n)^{n-1} \)).

We now find an explicit solution to \( F = 0 \). If \( \{e_1, \ldots, e_n\} \) denotes the canonical basis of vectors in \( \mathbb{R}^n \), we define the vectors \( v_1, \ldots, v_n \) by \( v_1 = \sum_{k=1}^n e_k \), and for \( j = 2, \ldots, n \),

\[
v_j = v_1 - \sum_{\ell=0}^{j-2} 2e_{n-\ell}.
\]

By construction, the set \( \{v_1, \ldots, v_n\} \) is linearly independent, and so the \( n \times n \) matrix \( \mathcal{I} \), whose \( j \)th column is the vector \( v_j^T \), is invertible.
Consider the following point in $V_\Phi \times V_\Psi$:

$$(\hat{\Phi}, \hat{\Psi}) = -\frac{\pi}{2} ((v_1, \ldots, v_{n-1}), v_n);$$

then it is easy to compute that

$$(8) \quad \tilde{P}(\hat{\Phi}, \hat{\Psi}) = i I,$$

where $\tilde{P}$ is as in (7). If we define

$$\hat{A} \equiv (\hat{a}_1, \ldots, \hat{a}_n)^T = I - \frac{1}{2} \omega^T,$$

then

$$F(\hat{\Phi}, \hat{\Psi}, \hat{A}, \omega) = 0.$$

Because the $\omega_j$ are rationally independent, it follows that the components $\hat{a}_k$ of $\hat{A}$ are all nonzero.

We now show that we can use the implicit function theorem at the point $(\hat{\Phi}, \hat{\Psi})$. Define the $n \times n$ invertible matrix $U_j$ to be the diagonal matrix whose $k$th diagonal element is the $k$th component of the vector $v_j$ (in particular, $U_1$ is the identity matrix). Note also that $U_j^{-1} = U_j$, $j = 1, \ldots, n$. We easily compute the derivative

$$J \equiv D_{(\Psi, A)} F(\hat{\Phi}, \hat{\Psi}, \hat{A}; \omega) = \begin{pmatrix} \hat{a}_n U_n & i I \\ \hat{a}_n U_n & -i I \end{pmatrix},$$

which is invertible, and its inverse is easily computed as

$$J^{-1} = \begin{pmatrix} \frac{1}{2} i U_n & \frac{1}{2} i U_n \\ -\frac{1}{2} I^{-1} & \frac{1}{2} I^{-1} \end{pmatrix}.$$

By the implicit function theorem, there exist a neighborhood $N$ of $\hat{\Phi}$ in $V_\Phi$ and a unique smooth function

$$\begin{align*}
G : N & \quad \mapsto \quad V_\Psi \times \mathbb{R}^n, \\
\Phi & \quad \mapsto \quad G(\Phi) = (G_\Psi(\Phi), G_A(\Phi))
\end{align*}$$

such that

$$G(\hat{\Phi}) = (\hat{\Psi}, \hat{A})$$

and

$$F(\Phi, G(\Phi); \omega) \equiv 0 \quad \forall \Phi \in N.$$

Recall that $O_k$ is the dense orbit generated by $\tau_k \omega \mod 2\pi$ on the $n$-torus $T^n$. Let $O_\Phi \subset V_\Phi$ be the direct product of the dense orbits $O_k$ for $k = 1, \ldots, n-1$ and $O_\Psi$ be the dense orbit in $V_\Psi$. From (9), if $\Phi \in O_\Phi$ and $\Psi = G_\Psi(\Phi) \in O_\Psi$, then $A = G_A(\Phi)$ yields a solution to the original system of equations (6). Thus, to complete the proof, it remains to show that there exists a point $\Phi \in O_\Phi$ which is mapped by $G_\Psi$ to a point $\Psi \in O_\Psi$. 
We begin by showing that $G_\Psi$ is regular at $\hat{\Phi}$. An easy calculation shows that

$$K \equiv D_\Phi F(\hat{\Phi}, \hat{\Psi}, \hat{A}; \omega) = \begin{pmatrix} \hat{a}_1 U_1 & \hat{a}_2 U_2 & \cdots & \hat{a}_{n-1} U_{n-1} \\ \hat{a}_1 U_1 & \hat{a}_2 U_2 & \cdots & \hat{a}_{n-1} U_{n-1} \end{pmatrix},$$

and implicit differentiation of (10) yields that

$$DG(\hat{\Phi}) = \begin{pmatrix} DG_\Psi(\hat{\Phi}) \\ DG_A(\hat{\Phi}) \end{pmatrix} = -J^{-1} K$$

(11)

where $0$ denotes the $n \times n$ zero matrix. Consequently,

$$DG_\Psi(\hat{\Phi}) = \begin{pmatrix} -\hat{a}_1 U_1 & -\hat{a}_2 U_2 & \cdots & -\hat{a}_{n-1} U_{n-1} \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

(12)

and it follows that the mapping $G_\Psi : N \rightarrow V_\Psi$ is regular at $\hat{\Phi}$.

The density of $O_k$ in $T^n$ for $k = 1, \ldots, n - 1$ implies that for every $\epsilon > 0$ there exists $(\tau_{1,\epsilon}, \ldots, \tau_{n-1,\epsilon})$ such that

$$\Phi^\epsilon = (\tau_{1,\epsilon} \omega, \ldots, \tau_{n-1,\epsilon} \omega) \bmod 2\pi$$

is in an $\epsilon$-neighborhood of $\hat{\Phi}$, and we define a small $(n - 1)$-dimensional surface in $V_\Phi$ based at $\Phi^\epsilon$ by

$$S_{\Phi^\epsilon}^h = \{(\tau_1 \omega, \ldots, \tau_{n-1} \omega) \bmod 2\pi | \tau_j \in (\tau_j, \epsilon - h, \tau_j, \epsilon + h)\}$$

with $\epsilon, h$ small enough so that $S_{\Phi^\epsilon}^h \subset N$. Note that this surface is generated by small nonempty open intervals of $O_k$ for $k = 1, \ldots, n - 1$.

We now show that the image of $S_{\Phi^\epsilon}^h$ by $G_\Psi$ has a nontrivial transversal intersection with $O_\Psi$. To do this, we consider the function

$$T : N \rightarrow \mathbb{R}$$

defined by

(13) $$T(\Phi) = \det \left( DG_\Psi(\Phi) \cdot W_1^T \ DG_\Psi(\Phi) \cdot W_2^T \ \cdots \ DG_\Psi(\Phi) \cdot W_{n-1}^T \ \omega^T \right),$$

where

$$W_j := \frac{d}{dx_j}(x_1 \omega, \ldots, x_j \omega, \ldots, x_{n-1} \omega)$$
for \( j = 1, \ldots, n - 1 \) are \( n - 1 \) linearly independent vectors in \((\mathbb{R}^n)^{n-1}\). Obviously, \( T \) is continuous, and
\[
T(\hat{\Phi}) = \det \left( \begin{array}{ccc}
\hat{a}_1 & U_1 & \omega^T \\
\hat{a}_2 & U_2 & \omega^T \\
\vdots & \vdots & \vdots \\
\hat{a}_{n-1} & U_{n-1} & \omega^T \\
\hat{a}_n & U_n & \omega^T \\
\end{array} \right)
\]
\[
= \frac{(-1)^{n-1}}{\hat{a}_n^n} \det U_n \left( \begin{array}{ccc}
\hat{a}_1 & \omega^T \\
\hat{a}_2 & \omega^T \\
\vdots & \vdots \\
\hat{a}_{n-1} & \omega^T \\
\end{array} \right)
\]
\[
= \frac{(\omega_1 \omega_2 \cdots \omega_n)(\hat{a}_1 \hat{a}_2 \cdots \hat{a}_{n-1})}{\hat{a}_n^n} \det I
\]
\[
\neq 0.
\]

It follows that there is a neighborhood \( N' \subset N \) in which \( T \neq 0 \). So, by choosing \( \epsilon, h \) small enough such that \( S_{\epsilon, h}^h \subset N' \), the image of \( S_{\epsilon, h}^h \) by \( G_\Psi \) is transverse to \( O_\psi \). The density of the orbit \( O_\psi \) in \( V_\Psi \) guarantees that there are infinitely many intersections with \( G_\Psi(S_{\epsilon, h}^h) \) near the point \( \hat{\Psi} = G_\Psi(\hat{\Phi}) \).

The next theorem shows that the previous realization result holds for open sets near solutions found in Theorem 2.1.

**Theorem 2.2.** Suppose \( \omega_1 > 0, \omega_2 > 0, \ldots, \omega_n > 0 \) are linearly independent over the rationals. There exist a neighborhood \( N' \subset N \) of \( \omega = (\omega_1, \ldots, \omega_n) \) in \( \mathbb{R}^n \) and a smooth mapping
\[
H : N \rightarrow \mathbb{R}^n \times \mathbb{R}^n,
\]
\[
\omega \mapsto H(\omega) = (\tau(\omega), A(\omega)) = ((\tau_1(\omega), \ldots, \tau_n(\omega)), (a_1(\omega), \ldots, a_n(\omega)))
\]
such that
\[
\sum_{k=1}^{n} a_k(\omega) e^{-i\omega_j \tau_k(\omega)} = i\omega_j, \quad j = 1, \ldots, n,
\]
(14)
\[
\sum_{k=1}^{n} a_k(\omega) e^{i\omega_j \tau_k(\omega)} = -i\omega_j, \quad j = 1, \ldots, n,
\]
for all \( \omega \in N' \).

**Proof.** We consider the system \( F = 0 \) given by (7). We have already shown in Theorem 2.1 that, for fixed \( \omega \) linearly independent over the rationals, there exist infinitely many solutions to \( F = 0 \). We again use an implicit function theorem argument combined with the density of irrational torus flows.

Consider the mapping
\[
Q : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n,
\]
\[
(\tau, A, \omega) \mapsto Q(\tau, A, \omega) = F((\tau_1, \ldots, \tau_{n-1}, \tau_n), (a_1, \ldots, a_n), \omega),
\]
where \( F \) is as in (7). Therefore,
\[
D_\tau Q(\tau, A, \omega) = D_{(\phi_1, \ldots, \phi_{n-1}, \psi)} F((\tau_1, \ldots, \tau_{n-1}, \tau_n), (a_1, \ldots, a_n), \omega).
\]

\[
\begin{pmatrix}
\omega^T & 0 & 0 & \cdots & 0 \\
0 & \omega^T & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \omega^T
\end{pmatrix}
\]
where each 0 in the matrix above is an \( n \)-dimensional zero column vector; and

\[
D_A Q(\tau, A, \omega) = D_A F((\tau_1 \omega, \ldots, \tau_{n-1} \omega), \tau_n \omega; A; \omega).
\]

Thus, we wish to show that the \( 2n \times 2n \) matrix

\[
\begin{pmatrix}
D_\tau Q(\tau, A, \omega) & D_A Q(\tau, A, \omega)
\end{pmatrix}
\]

is invertible at the solutions to (5) which have the form of Theorem 2.1.

For positive integers \( p \) and \( q \), let \( \text{Mat}_{p,q} \) denote the space of \( p \times q \) matrices. Consider the following mappings associated to (16), (17), and (18):

\[
\mathcal{R}_1 : V_\Phi \times V_\Psi \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \text{Mat}_{2n,n}
\]

defined by

\[
\mathcal{R}_1(\Phi, \Psi, A, \omega) = D_{(\Phi, \Psi)} F(\Phi, \Psi, A; \omega) \cdot 
\begin{pmatrix}
\omega^T & 0 & 0 & \cdots & 0 \\
0 & \omega^T & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \omega^T
\end{pmatrix},
\]

\[
\mathcal{R}_2 : V_\Phi \times V_\Psi \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \text{Mat}_{2n,n}
\]

defined by

\[
\mathcal{R}_2(\Phi, \Psi, A, \omega) = D_A F(\Phi, \Psi, A; \omega),
\]

and

\[
\mathcal{R} : V_\Phi \times V_\Psi \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \text{Mat}_{2n,2n}
\]

defined by

\[
\mathcal{R}(\Phi, \Psi, A, \omega) = \begin{pmatrix}
\mathcal{R}_1(\Phi, \Psi, A, \omega) & \mathcal{R}_2(\Phi, \Psi, A, \omega)
\end{pmatrix}.
\]

Now, a simple computation (similar to those done in the proof of Theorem 2.1) shows that

\[
\mathcal{R} \left( -\frac{\pi}{2}(v_1, \ldots, v_n), A, \omega \right) = \begin{pmatrix}
\mathcal{Z} & i \mathcal{I} \\
\mathcal{Z} & -i \mathcal{I}
\end{pmatrix},
\]

where

\[
\mathcal{Z} = \begin{pmatrix}
a_1 \mathcal{U}_1 \omega^T & a_2 \mathcal{U}_2 \omega^T & \cdots & a_n \mathcal{U}_n \omega^T
\end{pmatrix}.
\]

If none of the \( a_j \) vanish, then the \( n \times n \) matrix \( \mathcal{Z} \) is invertible, since its determinant is

\[
\det \mathcal{Z} = \prod_{j=1}^n a_j \omega_j \det \mathcal{I} \neq 0.
\]

Thus,

\[
\mathcal{R} \left( -\frac{\pi}{2}(v_1, \ldots, v_n), A, \omega \right)^{-1} = \begin{pmatrix}
\frac{1}{2} \mathcal{Z}^{-1} & \frac{i}{2} \mathcal{Z}^{-1} \\
-\frac{i}{2} \mathcal{I}^{-1} & \frac{1}{2} \mathcal{I}^{-1}
\end{pmatrix}.
\]

By continuity, there is thus a neighborhood \( \mathcal{N} \) of the point \( \mathcal{R} \left( -\frac{\pi}{2}(v_1, \ldots, v_n) \right) \) in \( V_\Phi \times V_\Psi \) in which \( \mathcal{R} \) is invertible. By Theorem 2.1, there are infinitely many solutions of \( Q = 0 \) (see (15)) in \( \mathcal{N} \), and the Jacobian matrix (18) is thus invertible at these solutions. We get the conclusion of Theorem 2.2 by the implicit function theorem. ■
2.1. Example: $D_3$-symmetric system. Theorem 2.1 is written in the context of scalar DDEs. However, in this section, we look at an example of a $D_3$-symmetric system of DDEs where Theorem 2.1 can be applied and then proceed to explain the generalization of this theorem, which has applications to symmetric systems of DDEs.

**Example 2.3.** Let $\Gamma = D_3$, the group generated by $\kappa$ and $\gamma$, act on $\mathbb{R}^3$ as follows:

$\kappa(x_1, x_2, x_3) = (x_1, x_3, x_2), \quad \gamma(x_1, x_2, x_3) = (x_3, x_1, x_2)$.

Consider a linear $D_3$-symmetric coupled cell system with delayed coupling where each cell is one-dimensional and has the form

\[
\begin{align*}
\dot{x}_1 &= a_1 x_1(t - \tau_1) + a_2 [x_2(t - \tau_2) + x_3(t - \tau_2)], \\
\dot{x}_2 &= a_1 x_2(t - \tau_1) + a_2 [x_3(t - \tau_2) + x_1(t - \tau_2)], \\
\dot{x}_3 &= a_1 x_3(t - \tau_1) + a_2 [x_1(t - \tau_2) + x_2(t - \tau_2)],
\end{align*}
\]

where $x_i \in \mathbb{R}$ for $i = 1, 2, 3$ and $a_1, a_2, a_3 \in \mathbb{R}$. The characteristic equation of system (19) is obtained by substituting $(x_1, x_2, x_3) = (w_1 e^{\lambda t}, w_2 e^{\lambda t}, w_3 e^{\lambda t})$ into the equations. We obtain after simplification

\[
\lambda w_1 = a_1 e^{-\lambda \tau_1} w_1 + a_2 e^{-\lambda \tau_2} [w_2 + w_3],
\]

\[
\lambda w_2 = a_1 e^{-\lambda \tau_1} w_2 + a_2 e^{-\lambda \tau_2} [w_3 + w_1],
\]

\[
\lambda w_3 = a_1 e^{-\lambda \tau_1} w_3 + a_2 e^{-\lambda \tau_2} [w_1 + w_2],
\]

and rearranging the terms we have

\[
(\lambda - a_1 e^{-\lambda \tau_1} I - a_2 e^{-\lambda \tau_2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}) \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0,
\]

where $I$ is the $3 \times 3$ identity matrix. Letting $\alpha = \lambda - a_1 e^{-\lambda \tau_1}$ and $\beta = -a_2 e^{-\lambda \tau_2}$ equation (20) becomes

\[
\begin{pmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0.
\]

Let

\[
\Delta(\lambda) = \begin{pmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{pmatrix}.
\]

We complexify $\mathbb{R}^3$ and look at the isotypic decomposition of $\mathbb{C}^3$ by the action of $D_3$:

\[
\mathbb{C}^3 = V_0 \oplus V_1 \oplus V_2,
\]

where $V_0$ is the trivial representation of $D_3$ and $V_1, V_2$ are the standard irreducible representations of $D_3$ (all representations are one-dimensional complex). A basis for $V_0$ is $u_0 = (v, v, v)^t$. 
a basis for \( V_1 \) is \( u_1 = (v, e^{2\pi i/3}v, e^{4\pi i/3}v)^t \), and a basis for \( V_2 \) is \( u_2 = (v, e^{4\pi i/3}v, e^{2\pi i/3}v)^t \). Therefore,
\[
\Delta(\lambda)u_0 = (\alpha + 2\beta)u_0
\]
and
\[
\Delta(\lambda)u_1 = (\alpha - \beta)u_1, \quad \Delta(\lambda)u_2 = (\alpha - \beta)u_2
\]
since \( e^{4\pi i/3} = e^{2\pi i/3} \). Therefore, in the basis given by the isotypic decomposition of \( \mathbb{C}^3 \), \( \Delta(\lambda) \) block diagonalizes so that we have
\[
\begin{pmatrix}
\alpha + 2\beta & 0 & 0 \\
0 & \alpha - \beta & 0 \\
0 & 0 & \alpha - \beta
\end{pmatrix}
\begin{pmatrix}
\tilde{w}_1 \\
\tilde{w}_2 \\
\tilde{w}_3
\end{pmatrix} = 0.
\]

Hence, the eigenvalues are solutions to
\[
\det \Delta(\lambda) = (\alpha + 2\beta)(\alpha - \beta)^2 = (\lambda - a_1 e^{-\lambda \tau_1} - 2a_2 e^{-\lambda \tau_2})(\lambda - a_1 e^{-\lambda \tau_1} + a_2 e^{-\lambda \tau_2})^2 = 0.
\]

Each factor of the characteristic equation is of the same form as the characteristic equation for a scalar DDE. Therefore, by letting \( \tilde{a}_1 = a_1 \) and \( \tilde{a}_2 = 2a_2 \) in \( (\lambda - a_1 e^{-\lambda \tau_1} - 2a_2 e^{-\lambda \tau_2}) \), Theorem 2.1 applies directly. The same is true for the factor \( (\lambda - a_1 e^{-\lambda \tau_1} + a_2 e^{-\lambda \tau_2}) \), where we let \( \tilde{a}_1 = a_1 \) and \( \tilde{a}_2 = -a_2 \). Hence, for any choice of a set of complex numbers \( \Lambda = \{i\omega_1, i\omega_2\} \) with \( \omega_1, \omega_2 > 0 \) and rationally independent, there exists a linear \( D_3 \) symmetric coupled cell system including \( \Lambda \) in its spectrum.

In the context of bifurcation theory, the symmetry properties of the critical eigenspace depend on which factor contains the critical eigenvalue, and this leads to different bifurcation behavior. Two imaginary eigenvalues in the first factor correspond to a nonresonant Hopf/Hopf mode interaction (without symmetry), while the second case leads to a nonresonant \( D_3 \) Hopf/Hopf mode interaction. Details of the unfolding of these bifurcations can be found, respectively, in Kuznetsov [22] and Golubitsky, Stewart, and Schaeffer [14].

Note that Theorem 2.1 is not sufficient to guarantee the existence of a linear \( D_3 \)-symmetric coupled cell system with \( i\omega_1 \) satisfying the first factor and \( i\omega_2 \) satisfying the second factor simultaneously. We characterize this situation as follows. Let \( b_1^1 = b_2^2 = 1, \), \( b_1^2 = 2 \), and \( b_2^1 = -1 \), and for fixed rationally independent \( i\omega_1, i\omega_2 \) (with \( \omega_1, \omega_2 > 0 \)) we look for \( a_1, a_2 \) and \( \tau_1, \tau_2 \) such that
\[
(21)
\]

\[
\begin{align*}
& a_1 b_1^1 e^{-i\omega_1 \tau_1} + a_2 b_2^1 e^{-i\omega_1 \tau_2} = i\omega_1, \\
& a_1 b_1^2 e^{-i\omega_2 \tau_1} + a_2 b_2^2 e^{-i\omega_2 \tau_2} = i\omega_2,
\end{align*}
\]

and their complex conjugate equations are satisfied. This is the context of the next theorem, which is a generalization of Theorem 2.1. We state this result in a general form below and postpone the proof to section 4, as it follows similar steps as the proof of Theorem 2.1.

Note that in the proof of Theorem 2.1, the matrix \( I \) defined in (8) is nonsingular by construction, and this is a crucial step in the argument. For this more general result we shall present, the matrix which holds a similar role is denoted by \( I_B \) since it is a matrix consisting of \( \pm \) the constants \( b_k^l \) which appear in (21). The form of this matrix is not relevant for the
moment, and the structure of the matrix is described in section 4. We are now ready to state the theorem.

**Theorem 2.4.** Consider the factors

$$\prod_{j=1}^{r} \left( \lambda - \sum_{k=1}^{n} a_kb_k^j e^{-\lambda \tau_k} \right)$$

of a characteristic polynomial, where the constants \( b_k^j \in \mathbb{R} \setminus \{0\} \) are fixed for all \( j = 1, \ldots, r, \ k = 1, \ldots, n \), and suppose that \( \det \mathcal{I}_B \neq 0 \). Suppose that \( \omega_1^1, \omega_1^2, \ldots, \omega_1^r, \ldots, \omega_r^1, \ldots, \omega_r^2 \) are positive and linearly independent over the rationals where \( \ell_1 + \cdots + \ell_r = n \). Then there exist \( \tau_1 > 0, \tau_2 > 0, \ldots, \tau_n > 0, a_1 \in \mathbb{R}, a_2 \in \mathbb{R}, \ldots, a_n \in \mathbb{R} \) such that for all \( j = 1, \ldots, r, \)

$$\left( \lambda - \sum_{k=1}^{n} a_kb_k^j e^{-\lambda \tau_k} \right) = 0$$

has roots \( i\omega_k^j \) for \( \ell = 1, \ldots, \ell_j \).

This theorem is applied in the following section to the case of \( D_n \)-symmetrically coupled one-dimensional cell systems. If \( n \) odd, it is easy to show that \( b_k^j \neq 0 \) holds, but for \( n \) even, some of the \( b_k^j \)'s can be zero, and in those cases, Theorem 2.4 cannot be applied directly.

**Example 2.5.** Consider the case of a \( D_4 \)-symmetric ring of DDEs given by

$$\dot{x}_i = a_1x_i(t - \tau_1) + a_2[x_{i+1}(t - \tau_2) + x_{i-1}(t - \tau_2)],$$

where \( i = 1, \ldots, 4 \) and the indices are taken modulo 4. A calculation similar to the \( D_3 \) case above yields the characteristic equation

$$\det \Delta(\lambda) = (\lambda - a_1 e^{-\lambda \tau_1} - 2a_2 e^{-\lambda \tau_2})(\lambda - a_1 e^{-\lambda \tau_1})(\lambda - a_1 e^{-\lambda \tau_1} + 2a_2 e^{-\lambda \tau_2})^2 = 0.$$ 

Here Theorem 2.4 cannot be applied if we include the second factor of the characteristic equation since the \( b_k^j \) coefficient of \( a_2 \) is null. However, Theorem 2.4 can be applied if we are looking for critical eigenvalues distributed among the first and third factors.

### 3. Linear \( \Gamma \)-symmetric DDEs.

We now look at the case of \( \Gamma \)-equivariant linear retarded functional differential equations (RFDEs) depending on \( \ell \) discrete delays. For the results of this section, we find it convenient to introduce the well-known abstract setting (see, for instance, Hale and Verduyn-Lunel [20]), which is adapted to the symmetric case. Let \( C_n = C([-\tau, 0], \mathbb{C}^n) \) be the Banach space of continuous functions from the interval \([-\tau, 0]\), into \( \mathbb{C}^n \) \((\tau > 0)\) endowed with the norm of uniform convergence. Consider the linear homogeneous RFDE

$$\dot{z}(t) = \mathcal{L}_0(z_t),$$

(23)
where $L_0$ is a bounded linear operator from $C_n$ into $\mathbb{C}^n$. We write
\[
L_0(\varphi) = \int_{-\tau}^{0} d\eta(\theta)\varphi(\theta),
\]
where $\eta$ is an $n \times n$ matrix-valued function of bounded variation defined on $[-\tau,0]$. The characteristic equation is
\[
(24) \quad \det \Delta(\lambda) = 0, \quad \text{where} \quad \Delta(\lambda) = \lambda I_n - \int_{-\tau}^{0} d\eta(\theta)e^{\lambda \theta},
\]
where $I_n$ is the $n \times n$ identity matrix. Note that $e^{\lambda \theta} = e^{\lambda \theta}I_n$.

Suppose that $\Gamma$ is a compact group of transformations acting linearly on $C^n$. We say that (23) is $\Gamma$-equivariant if
\[
(25) \quad \gamma \cdot \eta(\theta) = \eta(\theta) \cdot \gamma \quad \forall \gamma \in \Gamma, \theta \in [-\tau,0].
\]
The group action of $\Gamma$ on $C^n$ induces an isotypic decomposition of $C^n$:
\[
C^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k,
\]
where $V_i = U_i \oplus \cdots U_i$ for irreducible representations $U_i$ of $\Gamma$ and $U_i \not\cong U_j$ for $i \neq j$. Since $\eta(\theta)$ commutes with the action of $\Gamma$, then
\[
\eta(\theta)V_i \subset V_i
\]
for all $i = 1, \ldots, k$.

Therefore, $\Delta(\lambda)$ also commutes with the representation of $\Gamma$. Indeed, for all $\gamma \in \Gamma$,
\[
\Delta(\lambda)\gamma = \lambda I \gamma - \left[\int_{-\tau}^{0} d\eta(\theta)e^{\lambda \theta}\right] \gamma \\
= \gamma \lambda I - \left[\int_{-\tau}^{0} d\eta(\theta)\gamma e^{\lambda \theta}\right] \\
= \gamma \lambda I - \left[\int_{-\tau}^{0} \gamma d\eta(\theta)e^{\lambda \theta}\right] \\
= \gamma \left(\lambda I - \int_{-\tau}^{0} d\eta(\theta)e^{\lambda \theta}\right) = \gamma \Delta(\lambda).
\]
Thus,
\[
\Delta(\lambda)V_i \subset V_i
\]
for all $i = 1, \ldots, k$, and in the orthogonal basis given by the isotypic decomposition, the matrix $\Delta(\lambda)$ block diagonalizes and we write
\[
\Delta(\lambda) = \text{diag}(\Delta_1(\lambda), \ldots, \Delta_k(\lambda)).
\]
The characteristic equation then becomes
\[
\det \Delta(\lambda) = \prod_{i=1}^{k} \det \Delta_i(\lambda).
\]
Therefore, we are led to the following result.

**Proposition 3.1.** Suppose that \( V_i = U_i \) and \( U_i \) is a one-dimensional irreducible representation of \( \Gamma \). Then

\[
\text{det} \Delta_i(\lambda) = \lambda - \sum_{j=1}^{\ell} a_j e^{-\lambda \tau_j}.
\]

**Corollary 3.2.** Theorem 2.1 applies to factors of the characteristic equation which correspond to the context of Proposition 3.1.

### 3.1. Delay coupled cell systems with \( D_n \)-symmetry, \( n \) odd

Our goal is to apply Theorem 2.4 to delay coupled cell systems with \( D_n \)-symmetry. We focus on the case of \( n \) odd because the assumption \( b_k^i \neq 0 \) is satisfied for all \( j, k \). As Example 2.5 shows, when \( n \) is even one cannot apply Theorem 2.4 in all cases because some coefficients \( b_k^i \) may be zero. Therefore, we perform the following analysis on the case in which \( n \) is odd only. Note that these computations are valid in the case in which \( n \) is even with minor modifications.

Multiple authors \([6, 10, 15, 16, 17, 24, 25, 29, 30]\) have studied Hopf bifurcation in \( D_n \)-symmetric rings of cells with delayed coupling where each cell is one-dimensional. The differential equation systems in those papers have the following general form. For \( i = 1, \ldots, n \), the dynamics of cell \( i \) is given, respectively, for \( n \) odd:

\[
\dot{x}_i(t) = f(X_i) + g(x_{i+1}, \ldots, x_{i+(n-1)/2}, x_{i-(n-1)/2}, \ldots, x_{i-1}),
\]

where \( X_i = (x_i(t-s_1), \ldots, x_i(t-s_m)), x_j = x_j(t-\tau_j) \) for \( j \neq i \), \( f : \mathbb{R}^n \to \mathbb{R}, g : \mathbb{R}^{n-1} \to \mathbb{R} \) are smooth functions, \( \tau_j, s_j \in [0, \tau] \) for all \( j \neq i \), and \( s = 1, \ldots, m \). Here, \( f \) is called the internal dynamics and \( g \) is the coupling function.

#### 3.1.1. Characterization of delayed \( D_n \) networks

We introduce a more general notation for delayed symmetrically coupled cell systems inspired by recent work on (not necessarily symmetric) coupled cell systems of ODEs; see Golubitsky and Stewart \([13]\) for a survey of the theory. Note that all the results of this section are straightforward generalizations of the nondelayed case.

Suppose that each cell in the system has phase space \( \mathbb{R}^k \). We generalize system (26) to

\[
\dot{\mathbf{X}}_i(t) = f(\mathbf{X}_i, \mathbf{X}_{i+1}, \ldots, \mathbf{X}_{i+(n-1)/2}, \mathbf{X}_{i-(n-1)/2}, \ldots, \mathbf{X}_{i-1}), \quad i = 1, \ldots, n,
\]

where

\[
\mathbf{X}_j = (X_j(t-\tau_1), \ldots, X_j(t-\tau_m)),
\]

\( f : (\mathbb{R}^{km})^n \to \mathbb{R}^k \) is a smooth function, and the position of \( \mathbf{X}_k \) corresponds to the coupling from cell \( k \) to cell \( i \) and \( \tau_j \geq 0 \) for \( j = 1, \ldots, m \). The following proposition is a straightforward consequence of the coupled cell system theory \([13]\). The proof is given for completeness.

**Proposition 3.3.** Any delay coupled network of \( n \)-odd identical cells depending on \( m \) delays can be written as (27).

**Proof.** Since the cells are identical, each cell has the same dimension \( k \) and the dynamics of all cells is given by the same function \( f \). The function \( f \) has arguments coming from every other cell in the network corresponding to possible connections from these other cells, and those depend possibly on the \( m \) delays by the definition of \( \mathbf{X}_j \). \( \blacksquare \)
We say that cells $j$ and $k$ have identical coupling to cell $i$ if

\begin{equation}
 f(X_i,\ldots,u,\ldots,v,\ldots) = f(X_i,\ldots,v,\ldots,u,\ldots),
\end{equation}

where $u$ and $v$ are permuted from positions $j$ and $k$. We rewrite system (27) as

\begin{equation}
 \dot{X} = F(\tilde{X}),
\end{equation}

where $X = (X_1,\ldots,X_n)^t$,

\[ \tilde{X} = \tilde{X}_i,\tilde{X}_{i+1},\ldots,\tilde{X}_{i+(n-1)/2},\tilde{X}_{i-(n-1)/2},\ldots,\tilde{X}_{i-1}, \]

and $F : (\mathbb{R}^{kn})^n \rightarrow \mathbb{R}^{nk}$ has the $i$th component given by the formulas above for $\dot{X}_i(t)$.

Consider the group $D_n$, with generators $\rho$ and $\kappa$ acting on $\mathbb{R}^{kn}$ as follows:

\begin{align}
 &\rho(X_1,\ldots,X_n) = (X_n,X_1,X_2,\ldots,X_{n-1}), \\
 &\kappa(X_1,\ldots,X_n) = (X_1,X_n,\ldots,X_{n+2-j},\ldots,X_{(n+3)/2},X_{(n+1)/2},\ldots,X_j,\ldots,X_2).
\end{align}

Without loss of generality we assume that our networks are transitive. That is, all cells in the network can be reached from any other cell via the coupling.

We now characterize the connections in the network so that the delay-differential system is $D_n$-symmetric. We think of each cell in the network as having $[(n-1)/2]$ neighbors on each side and an opposite cell if $n$ is even. Graphically, it is clear that an $n$-cell network is $D_n$-symmetric if for all cells in the network, all connections to and from the $j$th neighbor on each side (or the opposite cell if $n$ is even) are all the same; that is, the coupling term and its delay must be the same for all those connections. This idea is formalized in the next result.

**Proposition 3.4.** A transitive network of $n$ coupled identical cells with delays is $D_n$-equivariant if and only if it satisfies the conditions below:

(i) Suppose that cell 1 receives an input from cell $j$ with delay $\epsilon \in [0,\tau]$; then every cell $i$ in the network $(i = 2,\ldots,n)$ receives an input from cell $(i + j - 1) \bmod n$ with delay $\epsilon$ identical to the one received by cell 1.

(ii) For every connection in part (i), there is an identical connection from cell $i$ to cell $(i + j - 1) \bmod n$ with delay $\epsilon$.

**Proof.** The result is true for $n$ odd and even but we give the proof only for $n$ odd. We begin by looking at $\rho$-equivariance. Denote by $[w]_i$ the $i$th row of vector $w$. Then,

\[ [\rho F(\tilde{X})]_i = f(\tilde{X}_{i-1},\tilde{X}_i,\ldots,\tilde{X}_{i-2}), \]

and since $\rho X = (X_n,X_1,\ldots,X_{n-1})$ we have

\[ [F(\rho \tilde{X})]_i = f(\tilde{X}_{i-1},\tilde{X}_i,\ldots,\tilde{X}_{i-2}). \]

Thus, $\rho$-equivariance holds automatically by the structure of the equations.

Suppose that cell 1 receives an input from cell $j$. We look at the system of equations (27) and focus on the possible coupling from cell $(i + j - 1) \bmod n$ to cell $i$. Moreover, consider the possible connection from cell $i$ to cell $(i - j + 1) \bmod n$. Note that the connections from $(i + j - 1) \bmod n$ to $i$ and from $i$ to $(i - j + 1) \bmod n$ are obtained by taking the index and
subtracting \( j - 1 \). Finally, consider the possible connection from cell \((i - j + 1)\mod n\) to cell \(i\). We now show that \(F(\tilde{X})\) is \(\kappa\)-equivariant (and so \(D_n\)-equivariant) if and only if the connections defined above are identical. An easy computation shows that for all \(i = 1, \ldots, n\) we have

\[
[kF(\tilde{X})]_{n+2-i} = f(\tilde{X}_i, \ldots, \tilde{X}_{i+j-1}, \ldots, \tilde{X}_{i-j+1}, \ldots)
\]

and

\[
[F(k\tilde{X})]_{n+2-i} = f(\tilde{X}_i, \ldots, \tilde{X}_{i-j-1}, \ldots, \tilde{X}_{i+j-1}, \ldots),
\]

where \(n + 2 - i\) is taken modulo \(n\) for \(i = 1\).

We now show that parts (i) and (ii) imply \(\kappa\)-equivariance. If part (i) holds, the coupling from cell \((i + j - 1)\mod n\) to cell \(i\) and the coupling from cell \(i\) to cell \((i - j + 1)\mod n\) are identical. Then, by part (ii), the coupling from cell \((i - j + 1)\mod n\) to cell \(i\) is identical to the coupling from cell \(i\) to cell \((i - j + 1)\mod n\). Therefore, the coupling from cells \((i + j - 1)\mod n\) and \((i - j + 1)\mod n\) to \(i\) are identical. By definition of identical coupling given by (28) we have that

\[
f(\tilde{X}_i, \ldots, \tilde{X}_{i-j-1}, \ldots, \tilde{X}_{i+j-1}, \ldots) = f(\tilde{X}_i, \ldots, \tilde{X}_{i-j-1}, \ldots, \tilde{X}_{i+j-1}, \ldots).
\]

Since the dynamics of all cells is given by the same function \(f\), this is true for all \(i = 1, \ldots, n\). Thus, \(F\) is \(\kappa\)-equivariant.

Suppose now that \(F\) is \(\kappa\)-equivariant. Equality of both sides of the equivariance condition implies that for all \(i = 1, \ldots, n\), the couplings from cells \((i + j - 1)\mod n\) and \((i - j + 1)\mod n\) to \(i\) are identical. Since the dynamics of all cells is given by the same function \(f\), the coupling from cell \((i + j - 1)\mod n\) to cell \(i\) guarantees an identical coupling from cell \(i\) to cell \((i - j + 1)\mod n\), and this proves (i). But, the coupling from cell \((i - j + 1)\mod n\) to \(i\) is therefore identical to the coupling from cell \(i\) to cell \((i - j + 1)\mod n\). Hence there is an identical two-way coupling between cells \(i\) and \((i - j + 1)\mod n\), which proves (ii).

3.1.2. General form of the characteristic equation. We now focus our attention on delay coupled cell systems where each cell is one-dimensional, that is, \(k = 1\). The results of this section are also easy generalizations of the nondelayed case. We split the linear and nonlinear parts of system (27) and write the result in abstract form:

\[
\dot{X} = LX_t + H(X_t),
\]

where \(X_t \in C([-\tau,0],\mathbb{R}^n), \ L : C([-\tau,0],\mathbb{R}^n) \to \mathbb{R}^n\) is a bounded linear map, and \(H\) is a nonlinear mapping. Thus, \(L\) is \(D_n\)-equivariant, \(\eta(\theta)\) is an \(n \times n\) \(D_n\)-equivariant matrix of bounded variation, and

\[
L\phi = \int^0_{-\tau} d\eta(\theta)\phi.
\]

Proposition 3.5. The matrix \(\eta(\theta)\) is symmetric (\(\eta(\theta) = \eta(\theta)^T\)) with the following properties:

1. for all \(j = 1, \ldots, n\), \(\eta_{jj}(\theta) = p(\theta)\) for some function \(p\), and
2. for all \(i, k\) with \(i \neq k\), \(\eta_{kk}(\theta) = \eta_{(2+n-k)k}(\theta) = \eta_{kk}(\theta)\).
Proof. We use Proposition 3.4 to obtain information on $\eta$. By part (ii), the matrix $\eta(\theta)$ is symmetric. From the structure of (27), we deduce that for all $j = 1, \ldots, n$, $\eta_{ij}(\theta) = p(\theta)$ for some function $p(\theta)$. We denote by $\eta_{ij}(\theta)$ the element of $\eta$ corresponding to the coupling from cell $j$ to $i$. Consider $\eta_{j1}(\theta)$; then there is an identical connection from cell $j$ to cell $2 + n - j$ by part (i) and so $\eta_{j1}(\theta) = \eta_{1(2+n-j)}(\theta)$. By part (ii), the connection from cell $2 + n - j$ to cell 1 is identical to its reciprocal and so $\eta_{j1}(\theta) = \eta_{(2+n-j)1}(\theta)$. By part (i), we then have $\eta_{kl}(\theta) = \eta_{(2+n-k)l}(\theta) = \eta_{k1}(\theta)$ since the connections to cell $i$ are identical to the connections to cell 1.

Remark 3.6. This result can be obtained for higher-dimensional cells with a proof essentially similar to this one, but with more cumbersome notation. We decided to restrict ourselves to the one-dimensional case as this is the one which we study in detail in what follows.

The diagonalization of the linear equation is obtained using the results at the beginning of section 3 and are analogous to calculations for the $D_n$-symmetric ODEs found in Golubitsky, Stewart, and Schaeffer [14, Chapter XVIII]. The details are left to the reader. One obtains for $j = 0, \ldots, n - 1$

\begin{equation}
A_j(\theta) := p(\theta) + \sum_{k=2}^{(n+1)/2} 2 \cos(2\pi(k-1)j/n)\eta_{k1}(\theta).
\end{equation}

Note that $A_j(\theta) = A_{n-j}(\theta)$ for $j = 1, \ldots, [n/2]$. The block diagonalization of $\eta$ is given by the terms $A_j(\theta)$ for $j = 0, \ldots, n - 1$. Hence, in the basis given by the isotypic decomposition, we have

\[\Delta(\lambda) = \lambda I_n - \int_{-\tau}^{0} d\eta(\theta)e^{\lambda\theta} = \lambda I_n - \int_{-\tau}^{0} \text{diag}(dA_0(\theta)e^{\lambda\theta}, \ldots, dA_{n-1}(\theta)e^{\lambda\theta}).\]

Let $\Delta_j(\lambda) = \lambda - \int_{-\tau}^{0} dA_j(\theta)e^{\lambda\theta}$; then

\[\Delta(\lambda) = \text{diag}(\Delta_0(\lambda), \ldots, \Delta_{n-1}(\lambda)).\]

Therefore, the characteristic equation has the decomposition

\begin{equation}
\det \Delta(\lambda) = \det \Delta_0(\lambda) \prod_{j=1}^{(n-1)/2} [\det \Delta_j(\lambda)]^2 = 0.
\end{equation}

For completeness, the reader can verify that the corresponding formula for $n$ even is

\begin{equation}
\det \Delta(\lambda) = \det \Delta_0(\lambda) \det \Delta_{n/2}(\lambda) \prod_{j=1}^{n/2-1} [\det \Delta_j(\lambda)]^2 = 0,
\end{equation}

where the $A_j$ formula is slightly different from the one above.
3.1.3. Application of main theorems to the $D_n$ case. It is straightforward that Theorem 2.1 can be applied to any of the factors of the characteristic equations (32) or (33). Recall from Example 2.5 that Theorem 2.4 can possibly be applied in the case in which $n$ is even if the chosen factors of the characteristic equation satisfy $b^j_k \neq 0$ for all $j, k$. We do not pursue this case here.

To apply Theorem 2.4 in the $D_n$ case with $n$ odd, we need to verify that the coefficients $b^j_k$ in the factors of the characteristic equation are nonzero and that the nondegeneracy condition $\det \mathcal{I}_B \neq 0$ is satisfied. In fact, as shown in section 4, $\det \mathcal{I}_B \neq 0$ if and only if the matrix

$$
\mathcal{B} := \begin{bmatrix}
    b^1_1 & b^1_1 + \mu_1 & \cdots & b^1_1 + \mu_{r-1} \\
    b^2_1 & b^2_1 + \mu_1 & \cdots & b^2_1 + \mu_{r-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    b^r_1 & b^r_1 + \mu_1 & \cdots & b^r_1 + \mu_{r-1}
\end{bmatrix}
$$

is nonsingular, where $\ell_j$ is the number of imaginary eigenvalues satisfying the $j$th term of the product of the characteristic equation (22) and $\mu_j = \sum_{i=1}^{\ell_j} \ell_i$, where $\mu_r = n$ and $\mu_0 := 0$. Note that row $j$ of $\mathcal{B}$ contains coefficients belonging to the $j$th factor of the characteristic equation (22). We apply Theorem 2.4 to $D_n$-symmetric coupled cell systems depending on an arbitrary number of finite delays.

The characteristic equation is

$$
\det \Delta(\lambda) = \det \Delta_0(\lambda) \prod_{j=1}^{(n-1)/2} |\det \Delta_j(\lambda)|^2 = 0.
$$

We can write

$$
\Delta_j(\lambda) = \lambda - F(\lambda) - G_j(\lambda),
$$

where

$$
F(\lambda) = \sum_{i=1}^{p} a_i e^{-\lambda \tau_i}
$$

are the terms coming from the internal dynamics of each cell and

$$
G_j(\lambda) = \sum_{k=2}^{(n+1)/2} \left[ 2 \cos \left( \frac{2\pi(k-1)j}{n} \right) \right] \sum_{t=1}^{m_k} \alpha^k_t e^{-\lambda s^k_t}
$$

are the contributions from the coupling where $m_k$ is the number of delayed terms in the connection from cell $k$ to 1 and $\alpha^k_t$ are the respective coupling coefficients.

**Example 3.7.** As an example, consider a delay coupled $D_5$-symmetric cell. Let $u_s(\theta) = 0$ if $\theta = [-\tau, -s]$ and $u_s(\theta) = 1$ for $\theta \in (-s, 0]$, where $\tau \geq s$ for all delays $s$, and suppose

$$
\eta(\theta) = \begin{bmatrix}
    p(\theta) & \eta_{21}(\theta) & \eta_{31}(\theta) & \eta_{41}(\theta) & \eta_{51}(\theta) \\
    \eta_{21}(\theta) & p(\theta) & \eta_{31}(\theta) & \eta_{41}(\theta) & \eta_{51}(\theta) \\
    \eta_{31}(\theta) & \eta_{21}(\theta) & p(\theta) & \eta_{41}(\theta) & \eta_{51}(\theta) \\
    \eta_{41}(\theta) & \eta_{31}(\theta) & \eta_{21}(\theta) & p(\theta) & \eta_{51}(\theta) \\
    \eta_{51}(\theta) & \eta_{41}(\theta) & \eta_{31}(\theta) & \eta_{21}(\theta) & p(\theta)
\end{bmatrix},
$$
where
\[
p(\theta) = \sum_{i=1}^{2} a_i u_{\tau_i}(\theta), \quad \eta_{21}(\theta) = \sum_{\ell=1}^{3} \alpha_{\ell}^2 u_{s_\ell^2}(\theta), \quad \text{and} \quad \eta_{31}(\theta) = \sum_{\ell=1}^{2} \alpha_{\ell}^3 u_{s_\ell^3}(\theta)
\]
with the conditions \(\eta_{41}(\theta) = \eta_{31}(\theta)\) and \(\eta_{51}(\theta) = \eta_{21}(\theta)\) given by Proposition 3.5, part (2). Then,
\[
F(\lambda) = \sum_{i=1}^{2} a_i e^{-\lambda \tau_i}
\]
and
\[
G_j(\lambda) = \sum_{k=2}^{3} \left[ 2 \cos \left( \frac{2\pi(k-1)j}{n} \right) \right] \sum_{\ell=1}^{m_k} \alpha_{\ell}^k e^{\lambda \tau_{\ell}},
\]
where \(m_2 = 3\) and \(m_3 = 2\).

Thus, all coefficients \(b_j^k\) of \(I_B\) are nonzero and it is convenient to set \(b_j^1\) to be the coefficient of \(a_1\); that is, \(b_j^1 = 1\) for \(j = 0, 1, 2, \ldots, (n+1)/2\), and we keep this convention for the remainder of the paper.

We suppose that the characteristic equation \(\det \Delta(\lambda) = 0\) has purely imaginary roots coming from all factors; then for \(r = (n-1)/2\) we have
\[
B := \begin{bmatrix}
    b_1^1 & b_1^{1+\mu_1} & \cdots & b_1^{1+\mu_{r-1}} \\
    b_1^2 & b_1^{2+\mu_1} & \cdots & b_1^{2+\mu_{r-1}} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_1^r & b_1^{r+\mu_1} & \cdots & b_1^{r+\mu_{r-1}}
\end{bmatrix},
\]
and we assign the coefficients \(b_j^{1+\mu_j}\) as follows. We suppose that the first row corresponds to the factor for the trivial representation, which means that
\[
b_j^{1+\mu_j} = 2, \quad j = 1, 2, \ldots, (n-1)/2.
\]
Then, we set the remaining coefficients of each row to be equal to \(2 \cos \left( \frac{2\pi(k-1)j}{n} \right)\) for \(k = 2, 3, \ldots, (n+1)/2\), where row \(j+1\) has the coefficients of \(\Delta_j\) for \(j = 1, 2, \ldots, (n-1)/2\). This leads to the matrix
\[
B = \begin{bmatrix}
    1 & 2 & 2 & \cdots & 2 & 2 \\
    1 & 2 \cos \left( \frac{2\pi}{n} \right) & 2 \cos \left( \frac{4\pi}{n} \right) & \cdots & 2 \cos \left( \frac{(n-3)\pi}{n} \right) & 2 \cos \left( \frac{(n-1)\pi}{n} \right) \\
    1 & 2 \cos \left( \frac{4\pi}{n} \right) & 2 \cos \left( \frac{8\pi}{n} \right) & \cdots & 2 \cos \left( \frac{2(n-3)\pi}{n} \right) & 2 \cos \left( \frac{2(n-1)\pi}{n} \right) \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    1 & 2 \cos \left( \frac{(n-1)\pi}{n} \right) & 2 \cos \left( \frac{2(n-1)\pi}{n} \right) & \cdots & 2 \cos \left( \frac{(n-3)(n-1)\pi}{2n} \right) & 2 \cos \left( \frac{(n-1)^2\pi}{2n} \right)
\end{bmatrix}.
\]

Let \(i_1 < i_2 < \cdots < i_s\) be a set of indices chosen from \(\{0, \ldots, (n-1)/2\}\) defining a combination of factors from the characteristic equation (32). We now construct the \(s \times s\)
matrix $\mathcal{B}$ by removing rows and columns of (35) not in the set \{i_1, i_2, \ldots, i_s\}. Suppose that $i_1, \ldots, i_s$ are chosen from \{1, \ldots, (n-1)/2\}; then the matrix $\mathcal{B}$ is symmetric ($\mathcal{B}^T = \mathcal{B}$) and has the form

\begin{equation}
\mathcal{B} = \begin{bmatrix}
2 \cos \left( \frac{2\pi i_1^2}{n} \right) & 2 \cos \left( \frac{2\pi i_2}{n} \right) & \cdots & 2 \cos \left( \frac{2\pi i_{s-1}^2}{n} \right) & 2 \cos \left( \frac{2\pi i_s}{n} \right) \\
2 \cos \left( \frac{2\pi i_1 i_2}{n} \right) & 2 \cos \left( \frac{2\pi i_2^2}{n} \right) & \cdots & 2 \cos \left( \frac{2\pi i_{s-1} i_s}{n} \right) & 2 \cos \left( \frac{2\pi i_s^2}{n} \right) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
2 \cos \left( \frac{2\pi i_1 i_{s-1}}{n} \right) & 2 \cos \left( \frac{2\pi i_2 i_{s-1}}{n} \right) & \cdots & 2 \cos \left( \frac{2\pi i_{s-1}^2}{n} \right) & 2 \cos \left( \frac{2\pi i_s i_{s-1}}{n} \right) \\
2 \cos \left( \frac{2\pi i_1 i_s}{n} \right) & 2 \cos \left( \frac{2\pi i_2 i_s}{n} \right) & \cdots & 2 \cos \left( \frac{2\pi i_{s-1} i_s}{n} \right) & 2 \cos \left( \frac{2\pi i_s^2}{n} \right)
\end{bmatrix}.
\end{equation}

In the other case, $i_1 = 0$ and the matrix is of the form

\begin{equation}
\mathcal{B} = \begin{bmatrix}
1 & 2 & \cdots & 2 & 2 \\
1 & 2 \cos \left( \frac{2\pi i_2^2}{n} \right) & \cdots & 2 \cos \left( \frac{2\pi i_{s-1}^2 - 1}{n} \right) & 2 \cos \left( \frac{2\pi i_s i_2}{n} \right) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 2 \cos \left( \frac{2\pi i_{s-1}^2}{n} \right) & \cdots & 2 \cos \left( \frac{2\pi i_s^2 - 1}{n} \right) & 2 \cos \left( \frac{2\pi i_s i_{s-1}}{n} \right) \\
1 & 2 \cos \left( \frac{2\pi i_s^2}{n} \right) & \cdots & 2 \cos \left( \frac{2\pi i_{s-1}^2}{n} \right) & 2 \cos \left( \frac{2\pi i_s^2}{n} \right)
\end{bmatrix}.
\end{equation}

We can now state our result.

**Theorem 3.8.** Consider a linear $D_n$-symmetric coupled cell system with $n$ odd depending on $k$ delays $\tau_1, \ldots, \tau_k$, and let $i_1 < i_2 < \cdots < i_s$ be indices chosen from \{0, 1, 1, \ldots, (n-1)/2\} defining a combination of factors from the characteristic equation (32). We assume that the matrix $\mathcal{B}$ given by (36) or (37) is nonsingular. Suppose

$$
\omega_{\ell_1}, \ldots, \omega_{\ell_{i_1}}, \omega_{\ell_{i_2}}, \ldots, \omega_{\ell_{i_s}}, \ldots, \omega_{\ell_{i_s}}
$$

are positive and linearly independent over the rationals, where $\ell_{i_1} + \cdots + \ell_{i_s} = k$. Then there exist $\tau_1 > 0, \ldots, \tau_k > 0$ and real coefficients $a_{i}$ such that for all $m = 1, \ldots, s$

$$
det \Delta_{i_m}(\lambda) = 0
$$

has solutions $i\omega_{\ell}^m$ for $\ell = 1, \ldots, \ell_{i_m}$.

**Proof.** Since $n$ is odd, the coefficients

$$
b_k^j = 2 \cos \left( \frac{2\pi (k-1)j}{n} \right)
$$
are nonzero for all $k = 2, \ldots, (n + 1)/2$ and $j = 0, \ldots, (n - 1)/2$. Because $\mathcal{B}$ is assumed to be nonsingular, Theorem 2.4 applies and the result is obtained.

The condition that $\mathcal{B}$ is nonsingular does not always hold, as we show in the case $s = 2$. Consider the matrix (37) with $n = 9$ so that $i_2 \in \{1, 2, 3, 4\}$. Choosing $i_2 = 3$ we have the singular matrix

$$
\mathcal{B} = \begin{pmatrix}
1 & 2 \\
1 & 2
\end{pmatrix}.
$$

We now show that $\mathcal{B}$ is nonsingular in the case $s = 2$ if the matrix is given by (36); that is,

$$
\mathcal{B} = \begin{pmatrix}
2 \cos \left( \frac{2\pi i_1^2}{n} \right) & 2 \cos \left( \frac{2\pi i_1 i_2}{n} \right) \\
2 \cos \left( \frac{2\pi i_2 i_1}{n} \right) & 2 \cos \left( \frac{2\pi i_2^2}{n} \right)
\end{pmatrix}.
$$

We compute

$$
\det \mathcal{B} = 4 \left[ \cos \left( \frac{2\pi i_1^2}{n} \right) \cos \left( \frac{2\pi i_2^2}{n} \right) - \cos \left( \frac{2\pi i_1 i_2}{n} \right)^2 \right] = 2 \left[ \cos \left( \frac{2\pi (i_1^2 + i_2^2)}{n} \right) + \cos \left( \frac{2\pi (i_1^2 - i_2^2)}{n} \right) - \cos \left( \frac{4\pi i_1 i_2}{n} \right) - 1 \right].
$$

We show a few cases explicitly. First, the case $n = 3$ is not relevant since $i_1 < i_2$, $(n - 1)/2 = 1$, and $i_1 \neq 0$. We show the case $n = 5$, where we must have $i_1 = 1$ and $i_2 = 2$. This implies that $i_1^2 + i_2^2 = 5$, and so

$$
\det \mathcal{B} = 2 \left[ \cos \left( \frac{4\pi}{5} \right) - \cos \left( \frac{8\pi}{5} \right) \right] \neq 0.
$$

We now turn to the general case and show that the determinant cannot vanish. Because the three cosines are projections of $n$th roots of unity on the real axis for $n$ odd, then

$$
\cos \left( \frac{2\pi (i_1^2 + i_2^2)}{n} \right) + \cos \left( \frac{2\pi (i_1^2 - i_2^2)}{n} \right) - \cos \left( \frac{4\pi i_1 i_2}{n} \right) \neq 1.
$$

So if the determinant is to vanish, one of the cosines must be equal to 1. Since $i_1 < i_2$, there is only one option and we must have $i_1^2 + i_2^2 = n$. Thus, $i_1^2 = n - i_2^2$ and

$$
\cos \left( \frac{2\pi (i_1^2 - i_2^2)}{n} \right) = \cos \left( \frac{2\pi (n - 2i_2^2)}{n} \right) = \cos \left( \frac{4\pi i_2^2}{n} \right).
$$

If
\[
\cos \left( \frac{4\pi i_2^2}{n} \right) - \cos \left( \frac{4\pi i_1 i_2}{n} \right) = 0,
\]
this would imply \( i_1 = i_2 \), but we know that \( i_1 < i_2 \) and so \( \det \mathcal{B} \) cannot vanish. We summarize this result in the next theorem.

**Theorem 3.9.** Consider a linear \( D_n \)-symmetric coupled cell system with \( n \) odd depending on \( k \) delays \( \tau_1, \ldots, \tau_k \), and let \( i_1 < i_2 \) be indices chosen from \( \{1, \ldots, (n-1)/2\} \) defining a combination of factors from the characteristic equation (32). Suppose

\[
\omega^1_1, \ldots, \omega^1_{i_1}, \omega^2_1, \ldots, \omega^2_{i_2}
\]

are positive and linearly independent over the rationals, where \( i_1 + i_2 = k \). Then there exist \( \tau_1 > 0, \ldots, \tau_k > 0 \) and real coefficients \( a_1 \in \mathbb{R}, \ldots, a_p \in \mathbb{R} \) such that for \( m = 1 \) and \( m = 2 \)

\[
\det \Delta_{i_m}(\lambda) = 0
\]

has solutions \( i\omega^m_\ell \) for \( \ell = 1, \ldots, \ell_{i_m} \).

**4. Proof of Theorem 2.4.** Before we present the proof of Theorem 2.4, we describe in the next lemma the form of the matrix \( I_B \) which appears in the proof and compute its determinant.

**Lemma 4.1.** Let \( \ell_1, \ldots, \ell_r \) be positive integers and define \( \mu_j = \sum_{i=1}^j \ell_i \), where \( \mu_r = n \) and \( \mu_0 := 0 \). Consider the \( n \times n \) matrix

\[
I_B := [A_1 \cdots A_j \cdots A_r]^T,
\]

where

\[
A_j = \begin{bmatrix}
    b_1^j & \cdots & b^j_{1+\mu_j-1} & b^j_{1+\mu_j} & \cdots & b^j_{1+\mu_j+1} & \cdots & b^j_n \\
    b_1^j & \cdots & b^j_{1+\mu_j-1} & b^j_{1+\mu_j} & \cdots & -b^j_{\mu_j} & b^j_{\mu_j+1} & \cdots & b^j_n \\
    \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots & \vdots \\
    b_1^j & \cdots & b^j_{1+\mu_j-1} & b^j_{1+\mu_j} & \cdots & -b^j_{\mu_j} & b^j_{\mu_j+1} & \cdots & b^j_n
\end{bmatrix}
\]

is an \( \ell_j \times n \) matrix and all elements are nonzero. Then,

\[
\det I_B = \pm \prod_{j=1}^r \left[ (-2)^{\ell_j-1} \prod_{s=2}^{\ell_j} b^j_s \right] \det \mathcal{B},
\]

where

\[
\mathcal{B} := \begin{bmatrix}
b^1_1 & b^1_{1+\mu_1} & \cdots & b^1_{1+\mu_r-1} \\
b^2_1 & b^2_{1+\mu_1} & \cdots & b^2_{1+\mu_r-1} \\
\vdots & \vdots & \ddots & \vdots \\
b^r_1 & b^r_{1+\mu_1} & \cdots & b^r_{1+\mu_r-1}
\end{bmatrix}
\]
Proof. Substitute row $k$, denoted by $R_k$, of
\[ A_j = \begin{bmatrix} b^j_1 & \cdots & b^j_{\mu_j-1} & b^j_{1+\mu_j-1} & b^j_{2+\mu_j-1} & \cdots & b^j_{\mu_j} & b^j_{\mu_j+1} & \cdots & b^j_n \\ b^j_1 & \cdots & b^j_{\mu_j-1} & b^j_{1+\mu_j-1} & b^j_{2+\mu_j-1} & \cdots & -b^j_{\mu_j} & b^j_{\mu_j+1} & \cdots & b^j_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b^j_1 & \cdots & b^j_{\mu_j-1} & b^j_{1+\mu_j-1} & -b^j_{2+\mu_j-1} & \cdots & -b^j_{\mu_j} & b^j_{\mu_j+1} & \cdots & b^j_n \end{bmatrix} \]

for $k = 2, \ldots, \ell_j$ by $R_k - R_1$. The matrix $A_j$ becomes
\[ \tilde{A}_j := \begin{bmatrix} b^j_1 & \cdots & b^j_{\mu_j-1} & b^j_{1+\mu_j-1} & b^j_{2+\mu_j-1} & \cdots & b^j_{1+\mu_j} & b^j_{\mu_j} & b^j_{\mu_j+1} & \cdots & b^j_n \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -2b^j_{\mu_j} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & -2b^j_{1+\mu_j} & -2b^j_{\mu_j} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & -2b^j_{2+\mu_j-1} & \cdots & -2b^j_{1+\mu_j} & -2b^j_{\mu_j} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & -2b^j_{2+\mu_j-1} & \cdots & -2b^j_{1+\mu_j} & -2b^j_{\mu_j} & 0 & \cdots & 0 \end{bmatrix}. \]

We compute the determinant of $I_B$ by cofactor expansion starting with row 2 of $\tilde{A}_j$, which contains a unique nonzero element $-2b^j_{\mu_j}$. Denote by $C_{ij}$ the $(i, j)$-cofactor matrix. The row $2 + \mu_j - 1$ of $C_{(2+\mu_j-1), \mu_j}$ has a unique nonzero element $-2b^j_{1+\mu_j}$, and we perform a cofactor expansion along this row. The row $2 + \mu_j - 1$ of this new cofactor matrix also has a unique nonzero element $-2b^j_{2+\mu_j-1}$, and we proceed with the same process removing successively columns $3 + \mu_j - 1$ to $\mu_j$ (and the appropriate rows) until the cofactor matrix has only two rows corresponding to the original $\tilde{A}_j$ matrix and the second row has the unique nonzero element $-2b^j_{2+\mu_j-1}$ which is used to perform a cofactor expansion. Performing this process successively on each matrix $\tilde{A}_j$ for $j = 1, \ldots, r$, leaves as a cofactor matrix the $r \times r$ matrix $B$ defined in the statement. The formula in the lemma is written using $\mu_j = \mu_{j-1} + \ell_j$, and so the lemma is proved. □

We are now ready to prove our Theorem 2.4.

Proof of Theorem 2.4. The proof of this theorem is similar to the proof of Theorem 2.1. However, more notation is needed and the details of some calculations are more elaborate because we are now dealing with $r$ factors of the characteristic equation and the vector $\omega$ is separated in $r$ subvectors of possibly unequal length.

A necessary and sufficient condition for the conclusion of the theorem to hold is that the following algebraic system of $2n$ equations has a solution in the $2n$ unknowns $(\tau_1, \tau_2, \ldots, \tau_n)$,
\[ \begin{align*}
\sum_{k=1}^{n} a_k (b_k^1 e^{i \omega_1^k \tau_k}) &= i \omega_1^k, & \ell &= 1, \ldots, \ell_1, \\
\sum_{k=1}^{n} a_k (b_k^2 e^{i \omega_1^k \tau_k}) &= i \omega_2^k, & \ell &= 1, \ldots, \ell_2, \\
\vdots \\
\sum_{k=1}^{n} a_k (b_k^r e^{i \omega_1^k \tau_k}) &= i \omega_r^k, & \ell &= 1, \ldots, \ell_r,
\end{align*} \]

\[
\begin{align*}
\sum_{k=1}^{n} a_k (b_k^1 e^{i \omega_1^k \tau_k}) &= -i \omega_1^k, & k &= 1, \ldots, \ell_1, \\
\sum_{k=1}^{n} a_k (b_k^2 e^{i \omega_1^k \tau_k}) &= -i \omega_2^k, & \ell &= 1, \ldots, \ell_2, \\
\vdots \\
\sum_{k=1}^{n} a_k (b_k^r e^{i \omega_1^k \tau_k}) &= -i \omega_r^k, & \ell &= 1, \ldots, \ell_r.
\end{align*} \]

(38)

(39)

We introduce the following notation to describe the above system of equations in matrix form. Let

\[ \omega = (\omega_1^1, \ldots, \omega_1^{\ell_1}, \ldots, \omega_1^{\ell_r}, \ldots, \omega_r^1, \ldots, \omega_r^{\ell_r}), \]

and define (39) as

\[
\begin{pmatrix} P(\tau; \omega) \\ P(-\tau; \omega) \end{pmatrix} A^T = \begin{pmatrix} i \omega^T \\ -i \omega^T \end{pmatrix},
\]

where \( A = (a_1, \ldots, a_n) \), superscript \( T \) denotes transpose, and \( P(\tau; \omega) = P(\tau_1, \ldots, \tau_n; \omega) \) is the \( n \times n \) matrix of the form

\[
P(\tau; \omega) = \begin{bmatrix} P_1(\tau; \omega) \\ P_2(\tau; \omega) \\ \vdots \\ P_r(\tau; \omega) \end{bmatrix},
\]

whose entry at block \( j \), row \( \ell \), and column \( k \) is

\[ [P_j(\tau; \omega)]_{\ell k} = b_j^1 e^{-i \omega_1^k \tau_k}. \]

Note that \( P(\tau; \omega) = P(-\tau; \omega) \).

As in the proof of Theorem 2.1, we use the fact that since the \( \omega_1^k \) are rationally independent, then \( \omega \tau_k \) taken modulo \( 2\pi \) generates a dense orbit, denoted \( O_k \), on a torus \( T^n \).

Just as in Theorem 2.1, we embed the problem into a mapping \( F \) associated to (40). Let

\[
V = \{ \Phi = (\Phi^1, \ldots, \Phi^r) \mid \Phi^j = (\Phi_1^j, \ldots, \Phi_n^j), \Phi_1^j = (\varphi_1^{j_{1k}}, \ldots, \varphi_n^{j_{1k}}), j = 1, \ldots, r \},
\]
and define

\[ F : V \times \mathbb{R}^n \longrightarrow \mathbb{R}^{2n} \]

as

\[ F(\Phi^1, \ldots, \Phi^r, A; \omega) = \begin{pmatrix} \tilde{P}(\Phi^1, \ldots, \Phi^r) \\ \tilde{P}(-\Phi^1, \ldots, -\Phi^r) \end{pmatrix} \begin{pmatrix} A^T \end{pmatrix} - i \begin{pmatrix} \omega^T \\ -\omega^T \end{pmatrix}, \]

where \( A \) and \( \omega \) are as previously defined and

\[ \tilde{P}(\Phi^1, \ldots, \Phi^r) = \begin{bmatrix} \tilde{P}_1(\Phi^1) \\ \tilde{P}_2(\Phi^2) \\ \vdots \\ \tilde{P}_r(\Phi^r) \end{bmatrix} \]

with

\[ \begin{bmatrix} \tilde{P}_j(\Phi^1, \ldots, \Phi^n) \end{bmatrix}_{\ell k} = b^j_k e^{-i \varphi^j_{\ell k}} \]

for \( j = 1, \ldots, r \), \( \ell = 1, \ldots, \ell_j \), and \( k = 1, \ldots, n \). The definition of \( \tilde{P} \) uses the following coordinates of \( V \).

We single out some coordinates as follows to facilitate the use of the implicit function theorem. Let \( \Psi_j = \Phi^n_j \) and \( \Psi = (\Psi_1, \ldots, \Psi_r) \), and we now write

\[ \Phi = (\Phi^1_0, \ldots, \Phi^{n-1}_o), \]

where

\[ \Phi^j_o = (\Phi^1_j, \ldots, \Phi^{j-1}_j). \]

Thus, the mapping (41) can be written as \( F(\Phi, \Psi, A; \omega) \).

We now find an explicit solution of \( F = 0 \) using the vectors \( v_j \) defined in the proof of Theorem 2.1. Denote by \( T^\ell \) the (invertible) \( \ell \times \ell \) matrix whose \( j \)th column is the vector \( v^T_j \).

We define \( \mu_0 := 0 \), \( \mu_j := \sum_{i=1}^{j} \ell_i \), and

\[ \Theta_{\ell_j} := (\Phi^{1+\mu_j-1}_j, \ldots, \Phi^{\mu_j}_j). \]

We use the following base point in \( V \). For \( j = 1, \ldots, r \) define \( \hat{\Phi}_j \) be the point given by

\[ \hat{\Theta}_{\ell_j} = -\frac{\pi}{2} ((v_1, \ldots, v_{\ell_j-1}, v_{\ell_j}) \]

and \( \hat{\Phi}_i = -\frac{\pi}{2} v_i \) for \( i \notin \{\mu_1+\mu_j-1, \ldots, \mu_j\} \). In particular, \( \hat{\Phi}_{\mu_j} = -\frac{\pi}{2} v_{\ell_j} \).

We now evaluate \( \tilde{P}(\hat{\Phi}, \hat{\Psi}) \) by computing \( \tilde{P}_j(\hat{\Phi}, \hat{\Psi}) \) for \( j = 1, \ldots, r \):

\[ \tilde{P}_j(\hat{\Phi}, \hat{\Psi}) = i \begin{bmatrix} b^j_1 \cdots b^j_{\mu_j-1} b^j_{1+\mu_j-1} b^j_{2+\mu_j-1} \cdots b^j_{\mu_j} b^j_{\mu_j+1} \cdots b^j_n \\ b^j_1 \cdots b^j_{\mu_j-1} b^j_{1+\mu_j-1} b^j_{2+\mu_j-1} \cdots -b^j_{\mu_j} \mu_j+1 \cdots b^j_n \\ \vdots \cdots \vdots \cdots \cdots \vdots \cdots \vdots \cdots \vdots \cdots \vdots \cdots \vdots \\ b^j_1 \cdots b^j_{\mu_j-1} b^j_{1+\mu_j-1} -b^j_{2+\mu_j-1} \cdots -b^j_{\mu_j} \mu_j+1 \cdots b^j_n \end{bmatrix}. \]
Thus,
\[
\tilde{P}(\hat{\Phi}, \hat{\Psi}) = \begin{bmatrix}
\tilde{P}_1(\hat{\Phi}, \hat{\Psi}) \\
\vdots \\
\tilde{P}_j(\hat{\Phi}, \hat{\Psi}) \\
\vdots \\
\tilde{P}_r(\hat{\Phi}, \hat{\Psi})
\end{bmatrix}
:= iI_B,
\]
where $I_B$ is invertible by assumption. In particular, \(\tilde{P}(-\hat{\Phi}, -\hat{\Psi}) = -iI_B\). We define \(\hat{A}^T \equiv (\hat{a}_1, \ldots, \hat{a}_n)^T = I^{-1}B\omega^T\), which leads to the solution:
\[
F(\hat{\Phi}, \hat{\Psi}, \hat{A}; \omega) = 0.
\]
Because the $\omega_j$ are rationally independent, it follows that the components $\hat{a}_k$ of $\hat{A}$ are all nonzero.

We now show that we can use the implicit function theorem at the point found above. We define the $\ell \times \ell$ invertible matrix $U_j$ to be the diagonal matrix whose $k$th diagonal element is the $k$th component of the vector $v_j$ (in particular, $U_1$ is the identity matrix). Note also that $U_j^{-1} = U_j$, $j = 1, \ldots, r$. We compute
\[
J \equiv D_{(\Psi, A)}F(\hat{\Phi}, \hat{\Psi}, \hat{A}; \omega) = \begin{pmatrix}
\hat{U} & iI_B \\
\hat{U} & -iI_B
\end{pmatrix},
\]
where
\[
\hat{U} = \text{diag}(\hat{a}_n b_n^1 U_1^{1}, \ldots, \hat{a}_n b_n^{r-1} U_r^{r-1}, \hat{a}_n b_n^r U_r^r)
\]
is an $n \times n$ matrix with diagonal blocks of dimensions $\ell_1 \times \ell_1$ to $\ell_r \times \ell_r$. By the implicit function theorem, there exist a neighborhood $N$ of $\hat{\Phi}$ in $V_{\Phi}$ and a unique smooth function
\[
G : N \rightarrow V_{\Psi} \times \mathbb{R}^n,
G : \Phi \rightarrow G(\Phi) = (G_{\Psi}(\Phi), G_A(\Phi))
\]
such that
\[
G(\hat{\Phi}) = (\hat{\Psi}, \hat{A})
\]
and
\[
(42) \quad F(\Phi, G(\Phi); \omega) \equiv 0 \quad \forall \Phi \in N.
\]

Now that we have identified a set of solutions for $F = 0$, we wish to identify within this set solutions which lie on the dense orbits $O_k$.

We show now that $G_{\Psi}$ is regular at $\hat{\Phi}$. A computation shows that
\[
K \equiv D_{\Psi}F(\hat{\Phi}, \hat{\Psi}, \hat{A}; \omega) = \begin{pmatrix}
\hat{K} \\
\hat{K}
\end{pmatrix},
\]
where

$$\dot{K} = \text{diag}(\dot{K}_1, \ldots, \dot{K}_r)$$

is an \(n \times (n-1)n\) matrix where the block \(\dot{K}_j\) has dimensions \(\ell_j \times (n-1)\ell_j\) and is of the form

$$\dot{K}_j = \begin{pmatrix}
\dot{a}_1 b_1^j U_1^j & \cdots & \dot{a}_{1+\mu_j-1} b_1^{j+1-\mu_j} U_1^j & \dot{a}_{2+\mu_{j-1}} b_2^{j+1-\mu_j} U_2^j & \cdots & \dot{a}_{n-1} b_{n-1} U_{n-1}^j \\
\dot{a}_{1+\mu_j} b_1^{j+1-\mu_j} U_1^j & \cdots & \dot{a}_{n-1} b_{n-1} U_{n-1}^j
\end{pmatrix}.$$ 

The matrix \(J\) is invertible and its inverse is

$$J^{-1} = \begin{pmatrix}
\frac{1}{2} U^{-1}_1 & \frac{1}{2} U^{-1}_2 \\
-\frac{1}{2} I_B^{-1} & -\frac{1}{2} I_B^{-1}
\end{pmatrix}.$$ 

Implicit differentiation of (42) yields

$$DG(\dot{\Phi}) = \begin{pmatrix}
DG_{\Psi}(\dot{\Phi}) \\
DG_{A^r}(\dot{\Phi})
\end{pmatrix} = -J^{-1}K$$

(43)

where \(*\) is not important for our purposes and the first component is an \(n \times (n-1)n\) matrix composed of \(n-1\) block matrices

$$\mathcal{M}_j = \begin{pmatrix}
-\frac{\dot{a}_1 b_1^j}{a_{\mu_j} b_{\mu_j}^j} (U_1^j)^2 & \cdots & -\frac{\dot{a}_{1+\mu_{j-1}} b_1^{j+1-\mu_j}}{a_{\mu_j} b_{\mu_j}^j} (U_1^j)^2 & -\frac{\dot{a}_{1+\mu_{j-1}} b_2^{j+1-\mu_j}}{a_{\mu_j} b_{\mu_j}^j} U_1^j U_2^j & \cdots \\
-\frac{\dot{a}_{1+\mu_j} b_1^{j+1-\mu_j}}{a_{\mu_j} b_{\mu_j}^j} U_1^j & \cdots & -\frac{\dot{a}_{1+\mu_j} b_2^{j+1-\mu_j}}{a_{\mu_j} b_{\mu_j}^j} (U_1^j)^2 & \cdots & -\frac{\dot{a}_{n-1} b_{n-1}}{a_{\mu_j} b_{\mu_j}^j} (U_1^j)^2
\end{pmatrix}$$

of dimension \(\ell_j \times (n-1)\ell_j\), where \(j = 1, \ldots, r-1\). Recall that \(\mu_r = n\), so that we have

$$\mathcal{M}_r = \begin{pmatrix}
-\frac{\dot{a}_1 b_r^j}{a_{\mu_r} b_{\mu_r}^j} U_1^r U_1^r & \cdots & -\frac{\dot{a}_{1+n-1} b_{1+n-1}}{a_{\mu_r} b_{\mu_r}^j} U_1^r U_1^r & -\frac{\dot{a}_{1+n-1} b_1^j}{a_{\mu_r} b_{\mu_r}^j} U_1^r U_2^r & \cdots \\
-\frac{\dot{a}_{1+n-1} b_1^j}{a_{\mu_r} b_{\mu_r}^j} U_1^r & \cdots & -\frac{\dot{a}_{1+n-1} b_{1+n-1}}{a_{\mu_r} b_{\mu_r}^j} U_1^r & -\frac{\dot{a}_{1+n-1} b_1^j}{a_{\mu_r} b_{\mu_r}^j} U_2^r & \cdots & -\frac{\dot{a}_{n-1} b_{n-1}}{a_{\mu_r} b_{\mu_r}^j} U_{n-1}^r U_{n-1}^r
\end{pmatrix}.$$ 

Consequently,

$$DG_{\Psi}(\dot{\Phi}) = \text{diag}(\mathcal{M}_1, \ldots, \mathcal{M}_r)$$

is nonsingular and it follows that the mapping

$$G_{\Psi} : N \rightarrow V_{\Psi}$$

is regular at \(\dot{\Phi}\).

From the density of \(\mathcal{O}_k\) in \(T^n\) for \(k = 1, \ldots, n-1\), we know that for every \(\epsilon > 0\) there exists \((\tau_1, \epsilon), \ldots, (\tau_{n-1}, \epsilon)\) such that

$$\Phi_\epsilon^* = (\tau_1, \epsilon^\omega), \ldots, (\tau_{n-1}, \epsilon^\omega) \mod 2\pi$$
is in an \( \epsilon \)-neighborhood of \( \hat{\Phi} \) in \( N \), and we define a small \((n - 1)\)-dimensional surface in \( V_{\Phi} \) based at \( \Phi^* \) by
\[
S_{\Phi^*} = \{ (\tau_1 \omega, \ldots, \tau_{n-1} \omega) \mod 2\pi \mid \tau_j \in (\tau_{j,\epsilon} - h, \tau_{j,\epsilon} + h) \}
\]
with \( \epsilon, h > 0 \) small enough so that \( S_{\Phi^*} \subset N \). We now show that the image of \( S_{\Phi^*} \) by \( G_{\Phi} \) has a nontrivial transversal intersection with \( O_{\Psi} \).

Consider the following \( n - 1 \) vectors in \((\mathbb{R}^\ell)^{n-1} \times (\mathbb{R}^\ell)^{n-1} \times \ldots \times (\mathbb{R}^\ell)^{n-1} \simeq (\mathbb{R}^n)^{n-1} : \)
\[
W_1 = (\omega^1, 0, \ldots, 0; \omega^2, 0, \ldots, 0; \ldots; \omega^r, 0, 0), \\
W_2 = (0, \omega^1, 0, \ldots, 0; \omega^2, 0, \ldots, 0; \ldots; 0, \omega^r, 0), \\
\vdots \\
W_{n-2} = (0, \ldots, \omega^1, 0, \ldots, 0; \omega^2, 0, \ldots, 0; \ldots; 0, \omega^r, 0), \\
W_{n-1} = (0, \ldots, 0, \omega^1; 0, \ldots, 0, \omega^2; \ldots; 0, 0, \omega^r),
\]
where 0 represents the 0 vector in the respective space \( \mathbb{R}^\ell \), and we recall that
\[
(\omega^1, \ldots, \omega^r) = (\omega^1_1, \ldots, \omega^1_{\ell_1}, \ldots, \omega^r_1, \ldots, \omega^r_{\ell_r}).
\]
The set \( \{W_1, \ldots, W_{n-1}\} \) is linearly independent. We consider the function
\[
T : N \longrightarrow \mathbb{R}
\]
defined by
\[
T(\Phi) = \det \left( DG_{\Phi} (\Phi) \cdot W_1^T \, DG_{\Phi} (\Phi) \cdot W_2^T \, \ldots \, DG_{\Phi} (\Phi) \cdot W_{n-1}^T \, \omega^T \right),
\]
and recalling that \((U_1^i)^2 = I\) for all \( j, i \) we compute
\[
T(\Phi) = \det \left( DG_{\Phi} (\Phi) \cdot W_1^T \, DG_{\Phi} (\Phi) \cdot W_2^T \, \ldots \, DG_{\Phi} (\Phi) \cdot W_{n-1}^T \, \omega^T \right) = \det (\alpha_{jk}),
\]
where \( j = 1, \ldots, r, \ k = 1, \ldots, n \). The elements of the matrix \( (\alpha_{jk}) \) are
\[
\alpha_{1k} = \begin{cases} \\
-\frac{\hat{a}_{1k} b_{1}}{a_{\mu_1} b_{\mu_1}} U_1^i U_1^k (\omega^1)^T, & k = 1, \ldots, \ell_1, \\
-\frac{\hat{a}_{1k} b_{1}}{a_{\mu_1} b_{\mu_1}} (U_1^k)^2(\omega^1)^T, & k = 1 + \ell_1, \ldots, n - 1, \\
\frac{\hat{a}_{\mu_1 \mu_{\ell_1}} b_{\mu_1}}{a_{\mu_1} b_{\mu_1}} (U_1^k)^2(\omega^1)^T, & k = n,
\end{cases}
\]
and for \( j = 2, \ldots, r - 1 \)
\[
\alpha_{jk} = \begin{cases} \\
-\frac{\hat{a}_{jk} b_{j}}{a_{\mu_j} b_{\mu_j}} (U_1^j)^2(\omega^j)^T, & k = 1, \ldots, \mu_{j-1}, \quad k = \mu_j + 1, \ldots, n - 1, \\
-\frac{\hat{a}_{jk} b_{j}}{a_{\mu_j} b_{\mu_j}} U_1^j U_1^k - \mu_{j-1}(\omega^j)^T, & k = 1 + \mu_{j-1}, \ldots, \mu_j, \\
\frac{\hat{a}_{\mu_j \mu_{\ell_j}} b_{\mu_j}}{a_{\mu_j} b_{\mu_j}} (U_1^j)^2(\omega^j)^T, & k = n,
\end{cases}
\]
and finally
\[ \alpha_{rk} = \begin{cases} 
\frac{\hat{a}_k b_{\mu r}^r}{\mu_{\mu r}} \mathcal{U}^r_k \mathcal{U}^r_1 (\omega) T, & k = 1, \ldots, \mu_r - 1, \\
\frac{\hat{a}_k b_{\mu r}^r}{\mu_{\mu r}} \mathcal{U}^r_k \mathcal{U}^r_{\mu r - 1} (\omega) T, & k = 1 + \mu_r - 1, \ldots, \mu_r - 1, \\
\frac{\hat{a}_k b_{\mu r}^r}{\mu_{\mu r}} (\mathcal{U}^r_{\mu r - 1})^2 (\omega) T, & k = n,
\end{cases} \]

where we recall that \( \mu_r = n \). Note that the elements of the last column are rewritten so as to lead to the significant simplification of the determinant to the following form:
\[ (47) \quad T(\hat{\Phi}) = \frac{(-1)^{n-1} \sigma_1 \cdots \sigma_{n-1}}{(\hat{a}_{\mu_1} b_{\mu_1}^r \cdots (\hat{a}_{\mu_r} b_{\mu_r}^r)^T)} \det(\text{diag}(\mathcal{U}_1^1, \mathcal{U}_1^2, \ldots, \mathcal{U}_1^{r-1}, \mathcal{U}_1^r)) \det \mathcal{I}_B, \]

where
\[ \mathcal{I}_B' = \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_r \end{pmatrix}, \]

and for \( j = 1, \ldots, r \)
\[ Q_j = \begin{bmatrix} b_{1j}^1 & \cdots & b_{1j}^{\mu_r - 1} & b_{2j}^{\mu_r + \mu_j - 1} & \cdots & b_{j}^{\mu_r} & b_{j+1}^{\mu_r + \mu_j - 1} & \cdots & \hat{a}_{\mu_j} b_{\mu_j}^r \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
b_{1j}^r & \cdots & b_{1j}^{\mu_j - 1} & b_{2j}^{\mu_r + \mu_j - 1} & \cdots & b_{j}^{\mu_j} & b_{j+1}^{\mu_r + \mu_j - 1} & \cdots & \hat{a}_{\mu_j} b_{\mu_j}^r \end{bmatrix} \]

is an \( \ell_j \times n \) matrix. Moreover, \( \det \mathcal{I}_B' \neq 0 \) since \( \det \mathcal{I}_B' = \hat{a}_n \det \mathcal{I}_B \) and \( \hat{a}_n \neq 0 \). Thus \( T(\hat{\Phi}) \neq 0 \), and the conclusion is exactly the same as in Theorem 2.1; hence the theorem is proved.

5. Conclusion. We have shown in this paper that \( n \) nonresonant eigenvalues on the imaginary axis can be realized by a scalar DDE with \( n \) delays. Moreover, the same is true for any collection of \( n \) imaginary eigenvalues in a neighborhood of an \( n \)-tuple of nonresonant imaginary eigenvalues. We have also shown how these results can be applied to nonscalar DDEs in the context of symmetric DDEs, where the characteristic equation decomposes according to the isotypic decomposition. We apply our result to delay coupled \( \mathbf{D}_n \)-symmetric cell systems with \( n \) odd.

There are several ways of extending the main result of our paper. One question we did not address in this paper is if \( n \) nonresonant nonzero imaginary eigenvalues constitute an upper bound for the realizability by a scalar equation with \( n \) delay. The case \( n = 1 \) is one such example since an easy calculation shows that we can have at most one imaginary eigenvalue on the imaginary axis. It is likely, but unknown, that this is also true for general \( n \).

One may want to study whether \( k \) zero eigenvalues in a single Jordan block and \( \ell \) nonresonant nonzero imaginary eigenvalues can be realized in a scalar DDE with \( k + \ell \) delays. This problem may be feasible by modifying the proof of Theorem 2.1 since the nonresonance of the \( \ell \) eigenvalues is again present. However, we expect the argument used in this paper to
break down for \( n \) nonzero imaginary eigenvalues with resonance. We can also study the same problem as in this paper but for higher-dimensional delay equations. One problem would be to find out if \( n \) nonresonant nonzero imaginary eigenvalues can be realized by an \( m \)-dimensional system with \( k \) delays. For instance, it is known that a pair of nonzero imaginary eigenvalues can be realized by a two-dimensional equation with one delay [8]. In this case \( n = 2, m = 2, k = 1, \) and so \( n = mk; \) is it possible to realize three nonresonant nonzero eigenvalues, or does the relationship \( n = mk \) provide a bound to realizability in general?

Another problem which can be studied is whether a restriction in the class of delay equation can change the realizability. For instance, the characteristic equation for a general two-dimensional system with one-delay \( \tau \) is

\[
\lambda^2 + a\lambda + b\lambda e^{-\lambda\tau} + c + de^{-\lambda\tau} = 0,
\]

while for a second-order equation with one delay in the feedback term we must set \( b = 0 \). We know in this case that two nonresonant nonzero imaginary eigenvalues can be realized by a second-order equation with a unique delay in the feedback term [4]. In fact, two imaginary eigenvalues with 1 : 2 resonance have been found in such an equation [5]. The obvious question is to see if \( n \) nonresonant (and resonant) nonzero imaginary eigenvalues can be realized within the class of \( n \)th-order scalar equations with one delay in the feedback term.

An extension of our main result in a direction relevant for studying bifurcations is to find out whether the \( n \) nonresonant nonzero imaginary eigenvalues can be realized by a scalar delay equation such that the remaining eigenvalues have negative real parts; that is, the multiple Hopf point lies at the boundary of the stability region for the equilibrium solution. This is typically verified in stability analysis of specific equations. For scalar problems see Bélair and Campbell [1] for a thorough analysis of a two-delay case and a review of several other cases. Yuan and Campbell [31] study the stability regions for \( D_n \)-symmetric rings of scalar cells with nearest neighbor delay coupling and simultaneously obtain the location of multiple Hopf bifurcation points at the boundary of the stability region.

Finally, let us mention the case of linear \( T \)-periodic equations with \( N + 1 \) delays:

\[
\dot{x} = \sum_{j=0}^{N} a_j(t)x(t - \tau_j),
\]

where each \( a_j(t) \) is a \( T \)-periodic \( n \times n \) matrix. Hale [19] states the following open problem:

*Is it possible to give a precise upper bound in terms of \( N \) on the number of Floquet multipliers of (48) that can have moduli 1?*

If we restrict (48) to scalar equations, we can pose a possibly simpler problem which is related to the main result of our paper:

*Is it possible to realize \( N + 1 \) complex numbers \( e^{\pm i\omega_1}, \ldots, e^{\pm i\omega_{N+1}} \) with \( \omega_1, \ldots, \omega_{N+1} \) positive and rationally independent as Floquet multipliers of the scalar equation (48)?*

This problem is automatically solved by Theorem 2.1 if a Floquet theorem can be applied to (48); that is, the Floquet exponents of the Floquet multipliers of (48) are eigenvalues of a scalar equation (1) with \( N + 1 \) delays. Such a theorem has not been proved in general; however, it may hold true given that some conditions are imposed on (48).
Acknowledgment. We would like to thank the referees for their suggestions concerning the contents and presentation of the paper.

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